# MATH 680A FALL 2016 GRADED HOMEWORK #4

*Write clearly, on separate paper. All questions carry equal weight.*
*You will receive credit for your three best answers.*

(1) Let $B$ be a bialgebra over a field $K$. Show that
$$C = \{c \in B \mid c^L \otimes c^R = c^R \otimes c^L\}$$
is a subalgebra of $(B, \nabla, \eta)$.

(2) For a commutative, unital ring $K$, consider the free module $K^2$ with standard basis $\{\mathbf{e}_1, \mathbf{e}_2\}$. Consider the endomorphism $\phi$ of $K^2 \otimes K^2$ which has matrix
$$\Phi = \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \alpha & \gamma & 0 \\ 0 & \gamma & \beta & 0 \\ 0 & 0 & 0 & \mu \end{bmatrix}$$
with respect to $\{\mathbf{e}_1 \otimes \mathbf{e}_1, \mathbf{e}_1 \otimes \mathbf{e}_2, \mathbf{e}_2 \otimes \mathbf{e}_1, \mathbf{e}_2 \otimes \mathbf{e}_2\}$. Determine necessary and sufficient conditions on the entries of the matrix $\Phi$ for the endomorphism $\phi\colon K^2 \otimes K^2 \to K^2 \otimes K^2$ to yield a solution of the braid equation
$$(\phi \otimes 1_{K^2})(1_{K^2} \otimes \phi)(\phi \otimes 1_{K^2}) = (1_{K^2} \otimes \phi)(\phi \otimes 1_{K^2})(1_{K^2} \otimes \phi)\,.$$

(3) For $1 < n \in \mathbb{Z}$, let $G$ be the cyclic group $\{g^j \mid 0 \le j < n\}$ of order $n$. Let $H$ be the usual complex group algebra $\mathbb{C}G$.
  (a) Show that $H$ constitutes a quasi-triangular Hopf algebra in conjunction with $1 \otimes 1$ as an $\mathcal{R}$-matrix.
  (b) Show that $H$ constitutes a quasi-triangular Hopf algebra in conjunction with
$$\frac{1}{n}\sum_{j=0}^{n-1}\sum_{k=0}^{n-1} \exp(-2\pi i jk/n) g^j \otimes g^k$$
as an $\mathcal{R}$-matrix.

(4) Let $f\colon U \to V$ be a linear transformation of finite-dimensional vector spaces over a field $K$. Show that the composite
$$V^* \xrightarrow{1_{V^*} \otimes \mathsf{coev}} V^* \otimes U \otimes U^* \xrightarrow{1_{V^*} \otimes f \otimes 1_{U^*}} V^* \otimes V \otimes U^* \xrightarrow{\mathsf{ev} \otimes 1_{U^*}} U^*$$
implements the dual linear transformation $f^*\colon V^* \to U^*$.