MATH 680A FALL 2016 GRADED HOMEWORK #3

Write clearly, on separate paper. All questions carry equal weight.
You will receive credit for your three best answers.

(1) Consider the rational vector space $\mathbb{Q}[X]$ of polynomials.

(a) Show that a coalgebra structure $(\mathbb{Q}[X], \Delta, \varepsilon)$ is defined uniquely by

$$\Delta\colon X^n \mapsto \sum_{k+l=n} X^k \otimes X^l\,, \quad \varepsilon\colon X^n \mapsto \delta_{0n}\,.$$

(b) Show that $(\mathbb{Q}[X], \Delta, \varepsilon)$ augments to a bialgebra with

$$\nabla\colon X^k \otimes X^l \mapsto \binom{k+l}{k} X^{k+l}\,, \quad \eta\colon 1 \mapsto 1\,.$$

(c) Show that there is an antipode $S\colon \mathbb{Q}[X] \to \mathbb{Q}[X]$ such that $(\mathbb{Q}[X], \nabla, \eta, \Delta, \varepsilon.S)$ is a Hopf algebra over $\mathbb{Q}$.

(2) Let H be a Hopf algebra over a field K. Suppose that M and N are left H-modules. Let $\underline{\underline{K}}(M,N)$ be the set of K-linear transformations from the vector space M to the vector space N. Show that the specification $(h \triangleright f)(m) := h^L \triangleright f(h^{RS} \triangleright m)$ for $h \in H$, $f \in \underline{\underline{K}}(M,N)$, and $m \in M$ makes $\underline{\underline{K}}(M,N)$ into a left H-module.

(3) Consider a coalgebra (C, Δ, ε) over a field K.

(a) Let V be a vector space over K. Show that $\Delta_{C\otimes V} := \Delta \otimes 1_V$ is the structure map of a left C-comodule on the space $C \otimes V$.

(b) Let M be a left C-comodule. Show that there is a vector space V such that M is a subcomodule of the comodule $C \otimes V$ given in (a).

(4) Let H be a bialgebra over a field K. Consider the left regular H-module structure $\nabla\colon H \otimes H \to H$ and H-comodule structure $\Delta\colon H \to H \otimes H$ on H.

Now let M be a vector space over K, carrying both a left H-module structure $\triangleright\colon H \otimes M \to M$ and a left H-comodule structure $\Delta_M\colon M \to H \otimes M$. Show that $\triangleright\colon H \otimes M \to M$ is a comodule homomorphism if and only if $\Delta_M\colon M \to H \otimes M$ is a module homomorphism.