MATH 680A FALL 2016 GRADED HOMEWORK #2

Write clearly, on separate paper. All questions carry equal weight.
You will receive credit for your three best answers.

(1) Consider a coalgebra (C, Δ, ε) over a field K. Let $\big(F(C), \nabla, \eta\big)$ be the tensor algebra over the underlying vector space C of the coalgebra.
  (a) Show that $\Delta\colon C \to C \otimes C$ and $\varepsilon\colon C \to K$ extend to yield a uniquely defined bialgebra $\big(F(C), \nabla, \eta, \Delta, \varepsilon\big)$.
  (b) Consider the *tensor Hopf algebra* $\big(T(C), \nabla, \eta, \Delta, \varepsilon, S\big)$ on the the underlying vector space C (Majid, p.7). What is the difference between the bialgebras $\big(T(C), \nabla, \eta, \Delta, \varepsilon\big)$ and $\big(F(C), \nabla, \eta, \Delta, \varepsilon\big)$?

(2) Let n be a positive integer. Let $\{x_i \mid 1 \le i \le n\}$ be a linearly independent subset of a coalgebra (C, Δ, ε) over a field. Suppose that there is a subset $S = \{x_{ij} \mid 1 \le i, j \le n\}$ of C such that $\Delta\colon x_j \mapsto \sum_{1 \le i \le n} x_i \otimes x_{ij}$ for $1 \le j \le n$. Show that

$$\Delta\colon x_{ij} \mapsto \sum_{1 \le h \le n} x_{ih} \otimes x_{hj} \quad \text{and} \quad \varepsilon\colon x_{ij} \mapsto \delta_{ij}$$

for $1 \le i, j \le n$.

(3) Consider the set $G = \{x \in B \mid \Delta\colon x \mapsto x \otimes x\,, \quad \varepsilon\colon x \mapsto 1\}$ of setlike elements of a bialgebra B over a field.
  (a) Show G forms a monoid under the multiplication of B.
  (b) If B is a Hopf algebra, show that G forms a group under the multiplication of B.

(4) Let H be a Hopf algebra over a field K.
  (a) Show that the set L of primitive elements of H forms a Lie algebra over K.
  (b) Let G be the group of setlike elements of H (as in the previous exercise). Show that

$$g\colon L \to L; l \mapsto g^{-1} l g$$

  for $g \in G$ provides a right action of G on L as a group of Lie algebra automorphisms.