MATH 680A FALL 2016 GRADED HOMEWORK #1

Write clearly, on separate paper. All questions carry equal weight.
You will receive credit for your three best answers.

(1) Let U and V be vector spaces over a field. Show that each element z of $U \otimes V$ may be written in the form

$$z = \sum_{i=1}^{r} x_i \otimes y_i$$

for a natural number r, a linearly independent subset $\{x_1, \dots, x_r\}$ of U, and a linearly independent subset $\{y_1, \dots, y_r\}$ of V.

(2) Let C be the real vector space consisting of solutions $y \colon \mathbb{R} \to \mathbb{R}$ of the differential equation $y'' + y = 0$. Show that

$$\begin{aligned} \Delta \colon\ & \cos \mapsto \cos \otimes \cos - \sin \otimes \sin, \\ & \sin \mapsto \cos \otimes \sin + \sin \otimes \cos \\ \varepsilon \colon\ & y \mapsto y(0) \end{aligned}$$

yields a coalgebra (C, Δ, ε).

(3) Consider the set

$$S = \{x \in C \mid \Delta \colon x \mapsto x \otimes x\,, \quad \varepsilon \colon x \mapsto 1\}$$

of setlike elements of a coalgebra C over a field. Show that S is linearly independent.

(4) Suppose that x is a non-zero primitive element within a Hopf algebra over $\mathbb{R}$. Prove, by induction on n, that $\{1, x, \dots, x^n\}$ is a linearly independent set for each natural number n.