MATH 617 FALL 2015 TAKE-HOME FINAL

No more than 5 answers will be graded: each worth 4 points, maximum 20.

(1) Let Ob(**Rel**) be the class of sets. For sets A and B, define **Rel**(A, B) to be the set of all subsets of $A \times B$. For $R \subseteq A \times B$ and $S \subseteq B \times C$, let

$$S \circ R = \{(a, c) \in A \times C \mid \exists b \in B \ . \ (a, b) \in R \text{ and } (b, c) \in S\} .$$

(a) Show that **Rel** is a category.

(b) Determine the product of two objects A, B of **Rel**.

(2) Let $\mathbf{Set}^{>1}$ be the full subcategory of the category of sets that comprises all sets with two or more elements. Show that $\mathbf{Set}^{>1}$ has no initial object and no terminal object.

(3) Let $\mathbf{C}$ be a category where each morphism is a monomorphism, and where there are two distinct morphisms having the same domain and the same codomain. Show that there are objects A and B of $\mathbf{C}$ for which the product $A \times B$ does not exist.

(4) Let $\mathbf{C}$ be the category of complex vector spaces, and let $\mathbf{R}$ be the category of real vector spaces. Let $G : \mathbf{C} \to \mathbf{R}$ be the forgetful functor that forgets the non-real scalar multiplications. Show that G has a left adjoint F.

(5) Consider the functor $S : \mathbf{Set} \to \mathbf{Set}$ with $SA = A \times A$ and

$$Sf : A \times A \to B \times B; (a, a') \mapsto (f(a), f(a'))$$

for a function $f : A \to B$. Show that S is naturally isomorphic to the functor $\mathbf{Set}(\mathbf{2}, \_\ )$, where $\mathbf{2} = \{0, 1\}$.

(6) Let $f \colon A \to B$ be a function.

(a) Show that there is a functor

$$f^{-1} \colon (\mathcal{P}(B), \subseteq) \to (\mathcal{P}(A), \subseteq)$$

of poset categories, with $f^{-1}(Y) = \{x \in A \mid f(x) \in Y\}$ for each subset Y of B.

(b) Show that f^{-1} has a right adjoint.