STRUCTURAL SPECIFICATIONS

Abstract. Initial objects as structural specifications. The design of special categories, such as slice categories and functor categories, for special purposes. Split extensions and modules.

1. Initial objects

1.1. Initial and terminal objects.

Definition 1.1. Let $\mathbf{C}$ be a category.

(a) An object $\bot$ of $\mathbf{C}$ is *initial* if $|\mathbf{C}(\bot, X)| = 1$ for all $X \in \mathbf{C}_0$.
(b) An object $\top$ of $\mathbf{C}$ is *terminal* if $|\mathbf{C}(X, \top)| = 1$ for all $X \in \mathbf{C}_0$.
(a) An object 0 of $\mathbf{C}$ is a *zero object* if it is both initial and terminal.

Example 1.2. (a) The empty set is initial in **Set**, while any singleton set is terminal.

(b) In the category of right modules over a ring S, the trivial S-module is a zero object.

(c) In a poset category, an initial object is a lower bound, and a terminal object is an upper bound. (Compare Definition 2.18, Categories.)

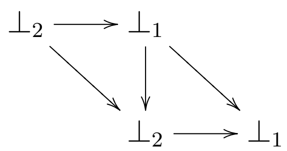
The concepts of initial and terminal objects are dual:

Proposition 1.3. *If $\bot$ is an initial object of a category $\mathbf{C}$, then it is a terminal object in $\mathbf{C}^{\mathsf{op}}$.*

Initial objects are specified uniquely up to isomorphism:

Proposition 1.4. *Any two initial objects of a category are isomorphic.*

Proof. Let $\bot_1$ and $\bot_2$ be initial objects in a category. Consider the following diagram



in which each arrow is uniquely specified since its domain is initial. The diagram commutes, since each path in the diagram starts from an initial object. The right hand triangle shows that $\bot_1 \to \bot_2$ is right invertible, while the left hand triangle shows that $\bot_1 \to \bot_2$ is left invertible. □

Corollary 1.5. *Any two terminal objects of a category are isomorphic, and any two zero objects of a category are isomorphic.*

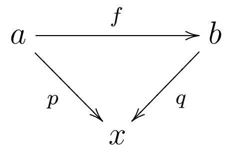
Proposition 1.4 means that mathematical objects may be specified uniquely up to isomorphism as initial objects of certain categories. For example, the ring of integers is uniquely specified as the initial object of the category of unital rings (Exercise 7).

1.2. **Slice categories and comma categories.** In order to apply the uniqueness of initial objects, it is necessary to design special categories.

1.2.1. *Slice categories.*

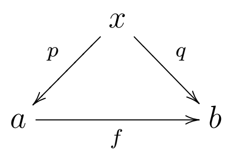
Definition 1.6. Let x be an object of a category C.

(a) The *slice category* C/x of C-*objects over* x has objects which are C-morphisms $p\colon a \to x$ with codomain x. A morphism in $(C/x)(p\colon a \to x, q\colon b \to x)$ is a C-morphism $f\colon a \to b$ such that the diagram



commutes. The identity at $p\colon a \to x$ is 1_a.

(b) The *slice category* x/C of C-*objects under* x has objects which are C-morphisms $p\colon x \to a$ with domain x. A morphism in $(x/C)(p\colon x \to a, q\colon x \to b)$ is a C-morphism $f\colon a \to b$ such that the diagram



commutes. The identity at $p\colon x \to a$ is 1_a.

Example 1.7. Let x be an object of a poset category $(X, \leq)$. Then X/x is the *down-set* $x^{\geq} = \{a \in X \mid x \geq a\}$, as a poset category with the order induced from $(X, \leq)$.

Example 1.8. Let $\top$ be a singleton set, a terminal object of **Set**. Then the slice category $\top/$**Set** is the category of *pointed sets*, sets with a selected element. Morphisms of pointed sets are functions which preserve the pointed element.

1.2.2. *Comma categories.* Comma categories provide a more flexible generalization of slice categories.

Definition 1.9. Consider functors $D \xrightarrow{S} C \xleftarrow{T} E$.

(a) The *comma category* $(S \downarrow T)$ has objects $(d, f : dS \to eT, e)$ in $D_0 \times C_1 \times E_0$, often abbreviated just to $f : dS \to eT$.
(b) An $(S \downarrow T)$-*morphism*

$$(g, h) : (d, f : dS \to eT, e) \to (d', f' : d'S \to e'T, e')$$

is an element (g, h) of $D_1(d, d') \times E_1(e, e')$ such that the diagram

$$\begin{array}{ccc} dS & \xrightarrow{gS} & d'S \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ eT & \xrightarrow[hT]{} & e'T \end{array} \tag{1.1}$$

in C commutes.

In many cases, one of the functors S or T in a comma category $(S \downarrow T)$ is taken to be a constant functor, whose domain is the category $\mathbf{1}$ with just one object 1 and one morphism 1_1.

Example 1.10. Let x be an object of a category C, with constant functor $[x] : \mathbf{1} \to C$.

(a) The slice category C/x is the comma category $(1_C \downarrow [x])$.
(b) The slice category x/C is the comma category $([x] \downarrow 1_C)$.

1.3. **Units in adjunctions.** Consider an adjunction

$$(F : D \to C, G : C \to D, \eta, \varepsilon)\,. \tag{1.2}$$

Proposition 1.11. *Let x be an object of D, with constant functor $[x] : \mathbf{1} \to D$. Then the component $\eta_x : x \to xFG$ is an initial object of the comma category $([x] \downarrow G)$.*

Proof. The comma category $([x] \downarrow G)$ starts with the functors

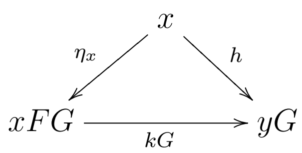
$$\mathbf{1} \xrightarrow{[x]} D \xleftarrow{G} C\,.$$

Let $h : x \to yG$ be an object of $([x] \downarrow G)$. Suppose that

$$(1_1, k) \in \mathbf{1}(1, 1) \times C(xF, y)$$

is a $([x] \downarrow G)$-morphism from $\eta_x \colon x \to xFG$ to $h \colon x \to yG$. Then the diagram

(1.3)



in D commutes, as an instance of the diagram (1.1). However, the adjunction isomorphism

$$C(xF, y) \cong D(x, yG)$$

sends $k \colon xF \to y$ to $\eta_x k^G$. By the commuting of (1.3), $\eta_x k^G = h$. Thus there is a unique C-morphism $k \colon xF \to y$ such that (1.3) commutes. Correspondingly, there is a unique $([x] \downarrow G)$-morphism $(1_1, k)$ from $\eta_x \colon x \to xFG$ to $f \colon x \to yG$. □

Dual to Proposition 1.11 is the following result. Its proof is relegated to Exercise 12.

Proposition 1.12. *Let y be an object of C, with constant functor $[y] \colon \mathbf{1} \to C$. Then the component $\varepsilon_y \colon yGF \to y$ is a terminal object of the comma category $(F \downarrow [y])$.*

Corollary 1.13. *Consider the adjunction* (1.2).

(a) *The unit $\eta \colon 1_D \to FG$ is uniquely determined by the functors F and G.*
(b) *The counit $\varepsilon \colon GF \to 1_C$ is uniquely determined by the functors F and G.*

Proof. (a) Apply Proposition 1.11 and the uniqueness of initial objects. Note that specification of the comma category $([x] \downarrow G)$ only involves the functor G.

(a) Apply Proposition 1.12 and the uniqueness of terminal objects. □

1.4. **Recognizing adjunctions.** Consider a functor $G \colon C \to D$. In §1.3, the existence of an adjunction

$$(F \colon D \to C, G \colon C \to D, \eta, \varepsilon)$$

implied that for each object x of D, the comma category $([x] \downarrow G)$ had an initial object. The converse result is the topic of this section. The idea of the proof mimics the discussion of Example 2.2 in the notes on Functors and Adjunctions.

Theorem 1.14. *Let $G\colon C \to D$ be a functor. Then G has a left adjoint if, for each object x of D, the comma category $([x] \downarrow G)$ has an initial object.*

Proof. For each object x of D, let $\eta_x\colon x \to xFG$ be the initial object of $([x] \downarrow G)$, where xF is an object in C mapping to the codomain of η_x under the object part of the functor G. This defines a function $F_0\colon D_0 \to C_0; x \mapsto xF$ which will become the object part of the functor $F\colon D \to C$.

Given a D-morphism $f\colon x' \to x$, the C-morphism $f^F\colon x'F \to xF$ is defined as the C-morphism yielding the unique $([x'] \downarrow G)$-morphism $(1_1, f^F)$ from $\eta'_x\colon x' \to x'FG$ to $f\eta_x\colon x' \to xFG$, i.e. making the diagrams

$$\begin{array}{ccc} & x' & \\ {\scriptstyle \eta'_x}\swarrow & & \searrow{\scriptstyle f\eta_x} \\ x'FG & \xrightarrow[fFG]{} & xFG \end{array}$$

or

$$\begin{array}{ccc} x' & \xrightarrow{\eta'_x} & x'FG \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle fFG} \\ x & \xrightarrow[\eta_x]{} & xFG \end{array} \tag{1.4}$$

in D commute. It is straightforward to verify that F is a functor (Exercise 13). The diagram (1.4) then shows that η is a natural transformation from 1_D to FG.

Now for x in D_0 and y in C_0, define

$$\varphi^x_y\colon C(xF, y) \to D(x, yG); g \mapsto \eta_x g^G \,.$$

For an element h of $D(x, yG)$, define $h\psi^y_x$ to be the unique element k of $C(xF, y)$ yielding an $([x] \downarrow G)$-morphism $(1_1, k)$ from $\eta_x\colon x \to xFG$ to $h\colon x \to yG$ — compare (1.3). Then φ^x_y and ψ^y_x are mutually inverse, so the desired isomorphism

$$C(xF, y) \cong D(x, yG)$$

is obtained. □

Corollary 1.15. *Let $F\colon D \to C$ be a functor. Then F has a right adjoint if, for each object y of C, the comma category $(C \downarrow [y])$ has a terminal object.*

Proof. This is dual to Theorem 1.14. □

2. Algebras in categories

2.1. **Categories with products.** Let $\mathbf{C}$ be a category. Definition 2.3 in the notes on Categories defined products $X_0 \times X_1$ of pairs of objects $X_0, X_1 \in \mathbf{C}_0$.

Definition 2.1. Suppose that $X_i \in \mathbf{C}_0$ for i in an index set I. Then the *product* $\prod_{i \in I} X_i$ is an object of $\mathbf{C}$ equipped with *projections*

$$\pi_i \colon \prod_{j \in I} X_j \to X_i$$

for each i in I, such that given $\mathbf{C}$-morphisms $f_i \colon Y \to X_i$ for $i \in I$, there is a unique morphism

$$\prod_{j \in I} f_j \colon Y \to \prod_{j \in I} X_j$$

such that $\left(\prod_{j \in I} f_j\right) \pi_i = f_i$ for $i \in I$.

Example 2.2. (a) If I is empty, the product is just a terminal object.

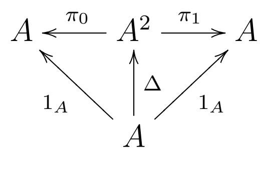
(b) Let $X = \{x_i \mid i \in I\}$ be a subset of $\mathbb{R}$ that is bounded below. Then in the poset category $(\mathbb{R}, \leq)$, the product $\prod_{j \in I} x_j$ is the infimum $\inf X$.

(c) If there is an object X such that $X_i = X$ for all $i \in I$, then the product $\prod_{i \in I} X_i$ is the I-th *power* X^I.

2.1.1. *Algebras in categories.* Suppose that $\mathbf{C}$ is a category in which the product $\prod_{i \in I} X_i$ exists whenever I is finite. (One describes $\mathbf{C}$ as a *category with finite products.*) Magmas (i.e., sets with a binary operation), semigroups, monoids, and groups are typically defined as objects in the category **Set** equipped with extra structure.

Definition 2.3. Let A be an object of $\mathbf{C}$.

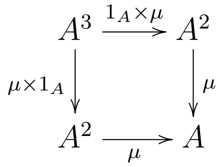
(a) The *diagonal* $\Delta \colon A \to A^2$ is defined by



using the product property of $A^2 = A \times A$.

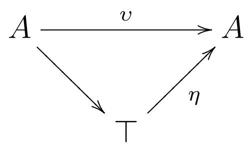
(b) If there is a morphism $\mu \colon A^2 \to A$, then A is a *magma* in $\mathbf{C}$.

(c) If further, the *associativity* diagram

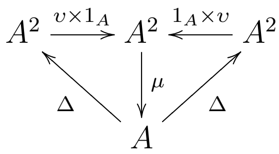


commutes, then A is a *semigroup* in $\mathbf{C}$.

(d) If further, there is a morphism $\eta\colon \top \to A$ with composite

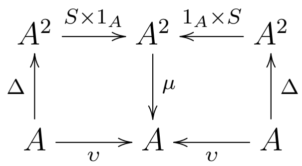


such that



commutes, then A is a *monoid* in $\mathbf{C}$.

(e) If, further, there is an *inversion* morphism $S\colon A \to A$ such that



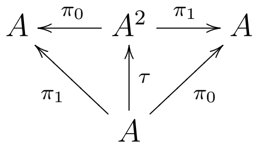
commutes, then A is a *group* in $\mathbf{C}$.

Remark 2.4. If $(A, \cdot, 1)$ is a group, then $\mu\colon (a_0, a_1) \mapsto a_0 \cdot a_1$ and $S\colon a \mapsto a^{-1}$, along with the selection of the identity element of A by $\eta\colon \top \to A$ (compare Exercise 2), make A a group in the category **Set** of sets.

Example 2.5. Let **Top** be the category consisting of topological spaces and continuous maps. A group in the category **Top** is a *topological group*.

2.1.2. *Commutative magmas.*

Definition 2.6. Let A be an object in a category $\mathbf{C}$ that has finite products. Then the *twist* is the map $\tau\colon A^2 \to A^2$ defined by the application

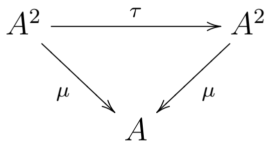


of the product property.

Example 2.7. If A is a set, then

$$\tau\colon A^2 \to A^2; (a_0, a_1) \mapsto (a_1, a_0)\,.$$

Definition 2.8. Let (A, μ) be a magma in a category $\mathbf{C}$ that has finite products. Then the magma is said to be *commutative* if the diagram

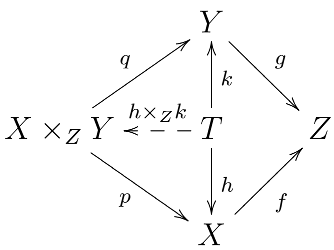


commutes. A commutative group A in a category is described as *abelian*, with *addition* $\mu\colon A^2 \to A$, *negation* $S\colon A \to A$, and *zero morphism* $\eta\colon \top \to A$.

2.2. **Pullbacks.**

Definition 2.9. Given arrows $f\colon X \to Z$ and $g\colon Y \to Z$ in a category $\mathbf{C}$, their *pullback* is an object $X \times_Z Y$ of $\mathbf{C}$, equipped with *projection* arrows $p\colon X \times_Z Y \to X$ and $q\colon X \times_Z Y \to Y$ satisfying $pf = qg$, such that for each object T of C and arrows $h\colon T \to X$, $k\colon T \to Y$ satisfying $hf = kg$, there is a unique arrow $h \times_Z k\colon T \to X \times_Z Y$ such that $(h \times_Z k)p = h$ and $(h \times_Z k)q = k$. The pullback is usually displayed as in the following diagram:

(2.1)



Example 2.10. In the category **Set** of sets, the pullback $X \times_Z Y$ is realized as $\{(x, y) \in X \times Y \mid xf = yg\}$, with the projections

$$p\colon X \times_Z Y \to X; (x, y) \mapsto x$$

and

$$q\colon X \times_Z Y \to Y; (x, y) \mapsto y\,.$$

In categories of algebras and homomorphisms, the same construction works.

The following proposition (with proof assigned as Exercise 18) shows how the pullback generalizes the product.

Proposition 2.11. *Suppose that X and Y are objects of a category $\mathbf{C}$ with terminal object $\top$. Then the pullback $X \times_\top Y$ is the product $X \times Y$.*

2.2.1. *Products in slice categories.* Let Q be an object of a category $\mathbf{C}$. If the category $\mathbf{C}$ has pullbacks, then the slice category $\mathbf{C}/Q$ has finite products: The product of two objects $p_0 : E_0 \to Q$ and $p_1 : E_1 \to Q$ is the pullback $p\colon E_0 \times_Q E_1 \to Q$ as in

$$\begin{array}{ccc} E_0 \times_Q E_1 & \xrightarrow{\pi_1} & E_1 \\ {\scriptstyle \pi_0}\downarrow & \searrow^{p} & \downarrow{\scriptstyle p_1} \\ E_0 & \xrightarrow[p_0]{} & Q \end{array} \tag{2.2}$$

with the composite morphism $p = \pi_0 p_0 = \pi_1 p_1$ to Q.

2.3. **Split extensions and modules.**

Definition 2.12. Let M be an abelian group. Then for a group G, the abelian group M is a (*right*) *G-module* if it is a right G-set whose representation $r\colon G \to M!$ restricts to $r\colon G \to \operatorname{End} M$.

Example 2.13. Let n be a positive integer.

(a) The abelian group $\mathbb{R}^n_1$ of n-dimensional row vectors is a right module for the *general linear group* $\mathsf{GL}(n, \mathbb{R})$ of invertible $n \times n$ real matrices.

(b) More generally, for any commutative unital ring S, the additive group S^n_1 is a right module over the group $(S^n_n)^*$ of invertible $n \times n$ matrices over S.

2.3.1. *Split extensions.*

Definition 2.14. Let M be a right module over a group Q. Then the *split extension* $E = Q \ltimes M$ is the set $Q \times M$ equipped with the product

$$(q_1, m_1)(q_2, m_2) = (q_1 q_2, m_1 q_2 + m_2)\,. \tag{2.3}$$

The split extension comes equipped with the projection

$$p : E \to Q; (q, m) \mapsto q \tag{2.4}$$

and the insertion η_Q or

$$\eta : Q \to E; q \mapsto (q, 0), \tag{2.5}$$

both of which are group homomorphisms (Exercise 20).

Example 2.15. For each positive integer n, the split extension

$$\mathsf{GA}(n,\mathbb{R}) = \mathsf{GL}(n,\mathbb{R}) \ltimes \mathbb{R}^n_1$$

is known as the *general affine group*. Compare Exercise 22.

Example 2.16. For each positive integer n, the *orthogonal group* $\mathsf{O}(n,\mathbb{R})$ is the subgroup $\{A \in \mathsf{GL}(n,\mathbb{R}) \mid AA^T = I_n\}$ of $\mathsf{GL}(n,\mathbb{R})$. Then the *Euclidean group* $\mathsf{E}(n,\mathbb{R})$ is the subgroup $\mathsf{O}(n,\mathbb{R}) \ltimes \mathbb{R}^n_1$ of the general affine group.

Let $\mathbf{Gp}$ denote the category of groups. If M is a module over a group Q, the projection $p : E \to Q$ of (2.4) is an abelian group in the slice category $\mathbf{Gp}/Q$. The addition (compare Definition 2.8) is

$$+ : E \times_Q E \to E; \big((q,m_1),(q,m_2)\big) \mapsto (q, m_1 + m_2)$$

and the zero morphism is given by the group homomorphism η of (2.5), determining the morphism

$$\begin{array}{ccc} Q & \xrightarrow{\eta} & E \\ {\scriptstyle 1_Q}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ Q & \xrightarrow[1_Q]{} & Q \end{array} \tag{2.6}$$

from the terminal object $1_Q : Q \to Q$ of the slice category $\mathbf{Gp}/Q$.

2.3.2. *Modules.* Given a module M over a group Q, the split extension $p : Q \ltimes M \to Q$ of (2.4) is an abelian group in the slice category $\mathbf{Gp}/Q$. For q in Q, the conjugation action of the element q^η on the normal subgroup $p^{-1}\{1\}$ of $Q \ltimes M$ is given by

$$(q,0)^{-1}(1,m)(q,0) = (1,mq), \tag{2.7}$$

thereby reflecting the action of Q on the module M.

Conversely, suppose that $p : E \to Q$ is an abelian group in the slice category $\mathbf{Gp}/Q$, with addition $+ : E \times_Q E \to E$ and zero morphism as in (2.6). Let M denote the inverse image $p^{-1}\{1\}$ of the identity element 1 of Q under p. For elements m_1 and m_2 of M, the pair (m_1, m_2) lies in the pullback $E \times_Q E$, and the image $m_1 + m_2$ of the pair (m_1, m_2) under the addition again lies in M. In this way, the set M receives an abelian group structure. In analogy with (2.7), each element q of Q acts on M by

$$q\colon m \mapsto (q^\eta)^{-1} m q^\eta,$$

making M a right Q-module.

Theorem 2.17. *Modules over a group Q are equivalent to abelian groups $p : E \to Q$ in the slice category $\mathbf{Gp}/Q$ of groups over Q.*

3. Exercises

(1) Show that the trivial group is a zero object in the category of groups.

(2) Suppose that a category C has a terminal object $\top$. Define a *point* of an object a of C to be a morphism $\top \to a$. Show that in **Set**, the points of an object A coincide with the elements of the set A.

(3) Let $\mathbf{C}$ be the category of all sets with at least two elements, and all functions between them. Show that $\mathbf{C}$ has no initial or terminal objects.

(4) Give an example of a poset category with no initial or terminal objects.

(5) Prove Proposition 1.3.

(6) Derive Corollary 1.5 from Proposition 1.4.

(7) Let S be a unital ring. Show that there is a unique unital ring homomorphism $\mathbb{Z} \to S$.

(8) Let x be an object of a category C.
   (a) Verify the category axioms for the slice category C/x.
   (b) Show that $(C^{\mathsf{op}}/x)^{\mathsf{op}} = x/C$.

(9) Suppose that $(X, \leq)$ is a poset category. Show that $(X, \leq)$ is isomorphic to the concrete category C with $C_0 = \{x^{\geq} \mid x \in X\}$ and $C(x^{\geq}, y^{\geq}) = \{x^{\geq} \hookrightarrow y^{\geq}\}$ for $x \leq y$ in X.

(10) Identify pointed sets as algebras with a single, nullary operation.

(11) Let **Ring** be the category of unital rings. Then an object $\varepsilon \colon R \to \mathbb{Z}$ of the slice category $\mathbf{Ring}/\mathbb{Z}$ is an *augmentation* of the ring R. Show that $p(X) \mapsto p(1)$ gives an augmentation of the integral polynomial ring $\mathbb{Z}[X]$.

(12) Prove Proposition 1.12.

(13) In the proof of Theorem 1.14, verify that F is a functor.

(14) Verify the claim of Example 2.2(b).

(15) Let $\mathbf{C}$ be a category with finite products.
   (a) Let $\mathbf{C}^2$ be the quiver with $(\mathbf{C}^2)_0 = \mathbf{C}_0 \times \mathbf{C}_0$ and
   $$\mathbf{C}^2\big((X,Y),(X',Y')\big) = \mathbf{C}(X,X') \times \mathbf{C}(Y,Y')$$
   for $X, X', Y, Y' \in \mathbf{C}_0$. Show that $\mathbf{C}^2$ is a category.
   (b) Let X_0 and X_1 be objects of $\mathbf{C}$. Let $\mathbf{C}\Delta/(X_0, X_1)$ be the subquiver of the slice category $\mathbf{C}^2/(X_0, X_1)$ whose object class consists of all objects of the form $(Y,Y) \to (X_0, X_1)$ for $Y \in \mathbf{C}_0$, and whose morphism class is formed by all morphisms $(f,f) \colon (Y,Y) \to (Z,Z)$ for $f \colon Y \to Z$ in $\mathbf{C}_1$. Show that $\mathbf{C}\Delta/(X_0, X_1)$ is a category.

(c) Show that $(\pi_0, \pi_1)\colon (X_0 \times X_1, X_0 \times X_1) \to (X_0, X_1)$ is a terminal object of $\mathbf{C}\Delta/(X_0, X_1)$.
(d) Conclude that products are uniquely specified up to isomorphism.

(16) Let $\mathbf{C}$ be a category with a terminal object $\top$.
(a) For $A \in \mathbf{C}_0$, show that the product $A \times \top$ exists in $\mathbf{C}$.
(b) Consider the functor $F_\top\colon \mathbf{C} \to \mathbf{C}; A \mapsto A \times \top$ (compare Proposition 2.3 in Functors and Adjunctions.) Show that there is a natural transformation $\lambda\colon 1_{\mathbf{C}} \to F_\top$ whose component

$$\lambda_A\colon A \to A \times \top$$

at each object A of $\mathbf{C}$ is an isomorphism.

(17) Show that a group in the category of groups is abelian.
(18) Prove Proposition 2.11.
(19) Verify the claims of §2.2.1.
(20) Check that the maps (2.4) and (2.5) are group homomorphisms. In particular, check that E is a group.
(21) In the context of Definition 2.14, define

$$r\colon E \to M! \tag{3.1}$$

by $m'(q, m)^r = m'q + m$ for $m', m \in M$ and $q \in Q$. Show that (3.1) gives a permutation action of E on M.

(22) Show that

$$\mathsf{GL}(n, \mathbb{R}) \ltimes \mathbb{R}^n_1 \to \mathsf{GL}(n+1, \mathbb{R}); (A, \mathbf{x}) \mapsto \begin{bmatrix} 1 & \mathbf{x} \\ & A \end{bmatrix}$$

provides an injective group homomorphism from the general affine group to the general linear group one dimension higher.

(23) Let S be the ring of integers modulo 2.
(a) What is the order of the group $(S^2_2)^* \ltimes S^2_1$ (compare with Example 2.13)?
(b) How does the group $(S^2_2)^* \ltimes S^2_1$ compare to the symmetric group of degree 4?

(24) Let $\theta\colon \mathbb{R}^2_1 \to \mathbb{R}^2_1$ be a transformation of the plane that preserves lengths. Show that $\theta \in \mathsf{E}(2, \mathbb{R})$, with the action of $\mathsf{E}(2, \mathbb{R})$ on the plane $\mathbb{R}^2_1$ given by (3.1).