MATH 505 SPRING 2015 PRACTICE FINAL

Each question is worth 6 points. The best 5 solutions will be taken.

(1) Can you find a single residue a modulo 7 with $\mathbb{Z}/_7(\sqrt{3}, \sqrt{5}\,) = \mathbb{Z}/_7(\sqrt{a}\,)$? Justify your answer.
(2) Let J_1 be the ideal $(X^3+X+1)\mathbb{Z}/_2[X]$ of the ring $\mathbb{Z}/_2[X]$. Let J_2 be the ideal $(X^3+X^2+X+1)\mathbb{Z}/_2[X]$. Show that the quotient rings $\big(\mathbb{Z}/_2[X]\big)\big/J_1$ and $\big(\mathbb{Z}/_2[X]\big)\big/J_2$ are not isomorphic.
(3) Determine all the irreducible cubic polynomials over $\mathsf{GF}(3)$.
(4) Let A be an abelian group. For each index set I, exhibit $\coprod_{i\in I} A$ as a subgroup of A^I.
(5) Let p be a prime number. Suppose that L is the lattice of positive integers under the divisibility relation. Let **Ring** be the category of unital ring homomorphisms. Consider the functor $F\colon L \to \textbf{Ring}$ sending integer n to $\mathsf{GF}(p^n)$. Show that each polynomial in $\mathsf{GF}(p)[X]$ has a root in $\varinjlim F$.
(6) Let V_4 be the Klein Vierergruppe. According to the Structure Theorem for Finitely Generated Abelian Groups, the abelian group

$$A = \big(\mathbb{Z}^2 \oplus \mathbb{Z}/_{10}\big) \otimes \big(\mathbb{Z} \oplus V_4\big)$$

is isomorphic to

$$\mathbb{Z}/J_1 \oplus \cdots \oplus \mathbb{Z}/J_l$$

for an ascending chain

$$J_1 \hookrightarrow \ldots \hookrightarrow J_l \tag{1}$$

of ideals of $\mathbb{Z}$. Determine the chain (1) of ideals.