MATH 505 SPRING 2015 GRADED HOMEWORK #3

Write clearly, on separate paper. All questions carry equal weight.
You will receive credit for your three best answers.

(1) Consider a positive integer n and a prime number p. Suppose that V is a vector space of dimension n over $\mathsf{GF}(p)$. Show that there is an invertible linear operator $\theta\colon V \to V$ of order $p^n - 1$.

(2) Let **Ring** be the category of unital ring homomorphisms. Let **Mon** be the category of monoid homomorphisms. Suppose that $G\colon \mathbf{Ring} \to \mathbf{Mon}$ is the forgetful functor which forgets the abelian group structure. Show that G has a left adjoint.

(3) Consider $\omega = \exp(2\pi i/5)$.
  (a) Show that there is a unique field E strictly intermediate between $\mathbb{Q}$ and $\mathbb{Q}(\omega)$.
  (b) Specify a basis for E as a vector space over $\mathbb{Q}$.

(4) Consider the Substitution Principle for commutative, unital rings:

> **Theorem:** Suppose that $\theta : R \to S$ is a unital ring homomorphism between commutative, unital rings R and S. For each fixed element c of the ring S, there is a unique unital ring homomorphism $\theta_c : R[X] \to S$, restricting to θ on R, with $\theta_c\colon X \mapsto c$.

Exhibit a category $\mathbf{C}$ such that $R[X]$ is an initial object of $\mathbf{C}$, and the Substitution Principle implements the statement that $R[X]$ is initial in $\mathbf{C}$.