MATH 505 SPRING 2015 GRADED HOMEWORK #2

(1) For an odd prime number p, consider $f(X)$ in $\mathbb{Z}/_p[X]$ with $\deg f(X) = 2$. Show that the function
$$f\colon \mathbb{Z}/_p \to \mathbb{Z}/_p; x \mapsto f(x)$$
is not bijective.

(2) (a) Show that the lattice $(\mathbb{R}, \leq)$ is distributive.
(b) Conclude that the lattice of subgroups of $\mathbb{Z}$ (under set-theoretic inclusion) is distributive. [Hint: You may use the Fundamental Theorem of Arithmetic.]

(3) Suppose that $(S, \vee)$ and $(S, \wedge)$ are semilattices. Suppose that
$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$
for all x, y, z in S.
(a) Show that
$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$
need not hold for all x, y, z in S.
(b) If $(S, \vee, \wedge)$ is a lattice, show that
$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$
holds for all x, y, z in S.

(4) Let F be a free group on a two-element set $\{x, y\}$. Let N be the smallest normal subgroup of F that contains the words x^2, y^2, and $(xy)^3$. Show that $|F/N| = 6$. [Hint: Consider the homogeneous space
$$N\backslash F = \{Ng \mid g \in F\}$$
with the actions
$$h\colon N\backslash F \to N\backslash F; Ng \mapsto Ngh$$
for h in F. In particular, consider the automaton $(N\backslash F, \{x, y\})$.]