

## MATH 505 SPRING 2015 GRADED HOMEWORK #1

*Write clearly, on separate paper. All questions carry equal weight.  
You will receive credit for your three best answers.*

- (1) Prove that  $\{1, \sqrt{3}, \sqrt{5}, \sqrt{15}\}$  is linearly independent over  $\mathbb{Q}$ .
- (2) Show that  $\{x \in \mathbb{C} \mid \dim_{\mathbb{Q}} \mathbb{Q}(x) < \infty\}$  forms a subfield of  $\mathbb{C}$ .
- (3) Let  $p$  be a prime number. Construct an example of a field  $F$  of characteristic  $p$  for which the Frobenius map  $\varphi : F \rightarrow F; x \mapsto x^p$  is not surjective. Justify your claims.
- (4) Prove that the following conditions on a field  $F$  are equivalent:
  - (a) Each nonconstant polynomial in  $F[X]$  has a root in  $F$ ;
  - (b) Each nonconstant polynomial in  $F[X]$  splits over  $F$ ;
  - (c) If  $E$  is an extension field of  $F$  for which  $\dim_F E$  is finite, then  $E = F$ .