LIMITS

Abstract. Limits and colimits. Applications.

1. Limits and colimits

1.1. **Limits.**

1.1.1. *Functor categories.*

Definition 1.1. Let D and C be categories.

(a) The class of all functors $F\colon D \to C$ is denoted by $\left(C^D\right)_0$.
(b) The class of all natural transformations $\tau\colon F \to G$ between objects of $\left(C^D\right)_0$ is denoted by $\left(C^D\right)_1$.
(c) Given an element $F\colon D \to C$ of $\left(C^D\right)_0$, the *identity natural transformation* 1_F has components $(1_F)_x = 1_{xF}\colon xF \to xF$ at each vertex x of D.
(d) Given natural transformations $\tau_1\colon F_0 \to F_1$ and $\tau_2\colon F_1 \to F_2$ between objects of $\left(C^D\right)_0$, the *composite* natural transformation $\tau_1\tau_2\colon F_0 \to F_2$ is defined by its components $(\tau_1\tau_2)_x = (\tau_1)_x(\tau_2)_x$ at each vertex x of D.
(e) Then $C^D = \left(\left(C^D\right)_0, \left(C^D\right)_1\right)$ is known as a *functor category*.

Example 1.2. Let G be a group that is realized as a category G with a single object X, and a morphism g for each group element g (compare Example 1.4, Categories). Consider the category $\underline{\underline{\mathbb{Z}}}$ of abelian groups. Then the functor category $\underline{\underline{\mathbb{Z}}}^G$ is the category of G-modules. Indeed, given a functor $r\colon G \to \underline{\underline{\mathbb{Z}}}$ with $Xr = M$, each morphism g^r (for $g \in G$) is an automorphism of the abelian group M, and the functoriality of r means that $mg^rh^r = m(gh)^r$ for $m \in M$ and $g, h \in G$.

Example 1.3. A poset is *discrete* if it has the form $(I, =)$ for a set I. Then for $\underline{2} = \{0, 1\}$ construed as a discrete poset category, the functor category $C^{\underline{2}}$ has an object class consisting of pairs (x_0, x_1) of objects of C, and a morphism class consisting of pairs $(f_0\colon x_0 \to y_0, f_1\colon x_1 \to y_1)$ of morphisms of C.

1.1.2. *Limits.*

Definition 1.4. Let J and $\mathbf{C}$ be categories, and let $F\colon J \to \mathbf{C}$ be a functor.

(a) The *diagonal functor* $\Delta\colon \mathbf{C} \to \mathbf{C}^J$ sends each object X of $\mathbf{C}$ to the constant functor $[X]\colon J \to \mathbf{C}$. Each morphism $f\colon X \to Y$ of $\mathbf{C}$ is sent to the natural transformation $[f]\colon [x] \to [y]$ with component $[f]_j = f$ at each object j of J.

(b) Let $[F]$ be the constant functor $[F]\colon \mathbf{1} \to \mathbf{C}$. Then the *limit* $\varprojlim F$ of F is a terminal object of the comma category $(\Delta \downarrow [F])$.

The comma category $(\Delta \downarrow [F])$ in Definition 1.4(b) is built on the pair

$$\mathbf{C} \xrightarrow{\Delta} \mathbf{C}^J \xleftarrow{[F]} \mathbf{1}$$

of functors whose common codomain is the functor category $\mathbf{C}^J$.

- An object $(Y, Y\Delta \to F, 1)$ of the comma category $(\Delta \downarrow [F])$ consists of a natural transformation $f\colon [Y] \to F$, specified by its components $f_j\colon Y \to jF$ at each object j of J.
- In particular, the terminal object $\varprojlim F$ of the comma category $(\Delta \downarrow [F])$ consists of an object of $\mathbf{C}$ itself, informally denoted as $\varprojlim F$, together with a natural transformation $\pi\colon [\varprojlim F] \to F$, specified by its components $\pi_j\colon \varprojlim F \to jF$ for each object j of J. These components are known as *projections*.
- The terminality of $\varprojlim F$ means that for each object Y of $\mathbf{C}$ with morphisms $f_j\colon Y \to jF$ for $j \in J_0$, there is a unique comma category morphism

$$(\varprojlim f, 1_1) \in \mathbf{C}(Y, \varprojlim F) \times \mathbf{1}(1,1)$$

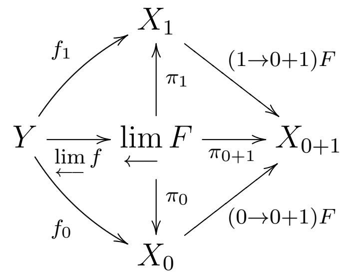
such that the diagrams

$$(1.1)\qquad \begin{array}{ccc} [Y] & \xrightarrow{[\varprojlim f]} & [\varprojlim F] \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \pi} \\ F & \xrightarrow[1_F]{} & F \end{array} \qquad \text{or} \qquad \begin{array}{ccc} Y & \xrightarrow{\varprojlim f} & \varprojlim F \\ & {\scriptstyle f_j}\searrow \quad \swarrow{\scriptstyle \pi_j} & \\ & jF & \end{array}$$

commute (in the second case, for each object j of J). (For the first diagram, compare (1.1) in Structural specifications).

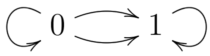
1.1.3. *Products and pullbacks as limits.* Products of pairs of objects are implemented as limits in which the index category J is the discrete poset category $\underline{2}$ of Example 1.3. More generally, an arbitrary product $\prod_{i \in I} X_i$ is implemented as a limit in which the index category J is the discrete poset category I given by the set I.

Example 1.5. Let J be the join semilattice poset category with Hasse diagram $0 \longrightarrow 0{+}1 \longleftarrow 1$. Then for a functor $F\colon J \to \mathbf{C}$ with $jF = X_j$ for $j \in J_0$, the limit $\varprojlim F$ is the pullback $X_0 \xleftarrow{\pi_0} X_0 \times_{X_{0+1}} X_1 \xrightarrow{\pi_1} X_1$. Note that $\pi_{0+1}\colon X_0 \times_{X_{0+1}} X_1 \to X_{0+1}$ is the composite $\pi_0(0 \to 0{+}1)^F = \pi_1(1 \to 0+1)^F$. The diagram

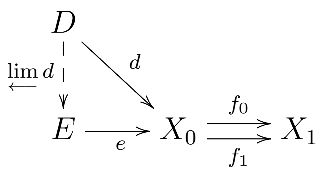


summarizes this case of the right-hand diagrams in (1.1).

1.1.4. *Equalizers.* Let $\|$ or *parallel* denote the category



with a parallel pair of non-identity arrows. Thus a functor $F\colon \| \to \mathbf{C}$ corresponds to a *parallel pair of arrows* $X_0 \underset{f_1}{\overset{f_0}{\rightrightarrows}} X_1$ in the category $\mathbf{C}$. The limit $\varprojlim F$ is a morphism $e\colon E \to X_0$ such that



commutes, i.e., such that $ef_0 = ef_1$, and such that $d\colon D \to X_0$ with $df_0 = df_1$ implies the existence of a unique morphism $\varprojlim d\colon D \to E$ with $(\varprojlim d)e = d$. The limit E is known as the *equalizer* of f_0 and f_1. In the category of sets, or concrete categories of algebras, one may take e as the insertion of the subset

$$E = \{x \in X_0 \mid xf_0 = xf_1\}$$

into X_0.

1.2. **Colimits.** Colimits are the duals of limits.

Definition 1.6. Let J and $\mathbf{C}$ be categories, and let $F\colon J \to \mathbf{C}$ be a functor, with $[F]$ as the constant functor $[F]\colon \mathbf{1} \to C$. Then the *colimit* $\varinjlim F$ of F is an initial object of the comma category $([F] \downarrow \Delta)$.

The comma category $([F] \downarrow \Delta)$ in Definition 1.6 is built on the pair

$$\mathbf{1} \xrightarrow{[F]} \mathbf{C}^J \xleftarrow{\Delta} \mathbf{C}$$

of functors whose common codomain is the functor category $\mathbf{C}^J$.

- An object $(1, F \to Y\Delta, Y)$ of the comma category $([F] \downarrow \Delta)$ consists of a natural transformation $f\colon F \to [Y]$, specified by its components $f_j\colon jF \to Y$ at each object j of J.
- In particular, the initial object $\varinjlim F$ of the comma category $([F] \downarrow \Delta)$ consists of an object of $\mathbf{C}$ itself, informally denoted as $\varinjlim F$, together with a natural transformation $\iota\colon F \to [\varinjlim F]$, specified by its components $\iota_j\colon jF \to \varinjlim F$ for each object j of J. These components are known as *insertions*.
- The initiality of $\varinjlim F$ means that for each object Y of $\mathbf{C}$ with morphisms $f_j\colon jF \to Y$ for $j \in J_0$, there is a unique comma category morphism

  $$(1_1, f) \in \mathbf{1}(1,1) \times \mathbf{C}(\varinjlim F, Y)$$

  such that the diagrams

$$\begin{array}{ccc} F & \xrightarrow{1_F} & F \\ {\scriptstyle\iota}\downarrow & & \downarrow{\scriptstyle f} \\ [\varinjlim F] & \xrightarrow[{[\varinjlim f]}]{} & [Y] \end{array} \qquad \text{or} \qquad \begin{array}{ccc} & jF & \\ {\scriptstyle\iota_j}\swarrow & & \searrow{\scriptstyle f_j} \\ \varinjlim F & \xrightarrow[\varinjlim f]{} & Y \end{array} \tag{1.2}$$

  commute (in the second case, for each object j of J).

Remark 1.7. Limits are also known as "projective limits" (because they come with projections) or "inverse limits," and may be written as "lim." Colimits, which may be written as "colim," are also known as "inductive limits" or "direct limits."

Remark 1.8. As a mnemonic for the direction of the arrow under limit and colimit symbols, one may think of the projections $jF \xleftarrow{\pi_j} \varprojlim F$ coming out of limits and the insertions $jF \xrightarrow{\iota_j} \varinjlim F$ going in to colimits.

1.2.1. *Coproducts.* Coproducts of pairs of objects are just colimits in which the index category J is the discrete poset category $\underline{2}$ considered in Example 1.3. More generally, an arbitrary coproduct $\coprod_{i\in I} X_i$ or $\sum_{i\in I} X_i$ is a colimit in which the index category J is the discrete poset category I given by the set I.

Example 1.9. (a) If I is empty, the coproduct $\coprod_{i\in I} X_i$ is just an initial object.

(b) Let $X = \{x_i \mid i \in I\}$ be a subset of $\mathbb{R}$ that is bounded above. Then in the poset category $(\mathbb{R}, \leq)$, the coproduct $\sum_{j\in I} x_j$ is the supremum $\sup X$.

(c) If there is an object X such that $X_i = X$ for all $i \in I$, then the coproduct $\sum_{i\in I} X_i$ is the I-th *multiple* IX.

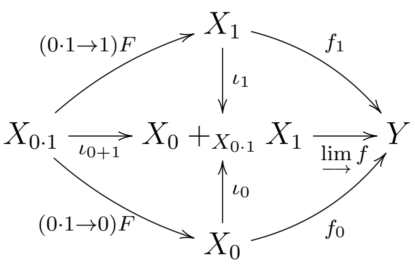
1.2.2. *Pushouts.*

Definition 1.10. Let J be the meet semilattice poset category with Hasse diagram $0 \longleftarrow 0\cdot 1 \longrightarrow 1$. Then for a functor $F\colon J \to \mathbf{C}$ with $jF = X_j$ for $j \in J_0$, the *pushout*

$$X_0 \xrightarrow{\iota_0} X_0 +_{X_{0\cdot 1}} X_1 \xleftarrow{\iota_1} X_1 \tag{1.3}$$

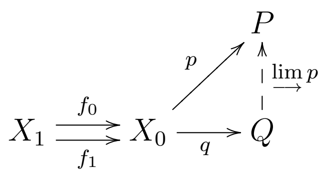
is the colimit $\varinjlim F$.

In the pushout (1.3), the composite $(0\cdot 1 \to 0)^F \iota_0 = (0\cdot 1 \to 1)^F \iota_1$ is the insertion $\iota_{0\cdot 1}\colon X_{0\cdot 1} \to X_0 +_{X_{0\cdot 1}} X_1$. The diagram



summarizes this case of the right-hand diagrams in (1.2).

1.2.3. *Coequalizers.* For a functor $F\colon \| \to \mathbf{C}$ yielding a parallel pair of arrows $X_1 \overset{f_0}{\underset{f_1}{\rightrightarrows}} X_0$ in a category $\mathbf{C}$, the colimit $\varinjlim F$ is known as a *coequalizer*. It is a morphism $q\colon X_0 \to Q$ such that



commutes, i.e., such that $f_0 q = f_1 q$, and such that $p\colon X_0 \to P$ with $f_0 p = f_1 p$ implies the existence of a unique morphism $\varinjlim p\colon Q \to P$ with $q(\varinjlim p) = p$.

2. Applications

Limit and colimits pervade mathematics.

2.1. **Coequalizers.**

2.1.1. *Equivalence relations.* Let X be a set, and let B be a binary relation on X. The projections

$$\pi_i \colon B \to X; (x_0, x_1) \mapsto x_i$$

for $i = 0, 1$ furnish a parallel pair $B \underset{\pi_1}{\overset{\pi_0}{\rightrightarrows}} X$ of arrows in the category **Set**. Let V be the equivalence relation *generated* by B, the smallest equivalence relation on X containing B. Then the natural projection

$$\text{nat}\, V \colon X \to X^V; x \mapsto x^V , \tag{2.1}$$

with $x^V = \{y \in X \mid x\ V\ y\}$ and $X^V = \{x^V \mid x \in X\}$, is the coequalizer of $B \underset{\pi_1}{\overset{\pi_0}{\rightrightarrows}} X$ in **Set** (Exercise 11).

2.1.2. *Group presentations.* A presentation

$$Q = \langle x_1, \dots, x_n \mid u_1 = v_1, \dots, u_r = v_r \rangle \tag{2.2}$$

of a group Q by generators $x_1, \dots, x_n$ subject to the relations

$$u_1 = v_1, \dots, u_r = v_r ,$$

where the u_i and v_i are elements of the free group F on the set

$$\{x_1, \dots, x_n\} ,$$

means that Q is being described as a coequalizer in the category **Gp** of groups. Specifically, let R be the free group on a set $\{y_1, \dots, y_r\}$. Define group homomorphisms

$$f_0 \colon R \to F; y_1 \mapsto u_1, \dots, y_r \mapsto u_r$$

and

$$f_1 \colon R \to F; y_1 \mapsto v_1, \dots, y_r \mapsto v_r$$

using the freeness of R on $\{y_1, \dots, y_r\}$. Then Q is the coequalizer $q \colon F \to Q$ of the parallel pair $R \underset{f_1}{\overset{f_0}{\rightrightarrows}} F$ in **Gp**. Indeed, $f_0 q = f_1 q$ implies that for $1 \le i \le r$, one has $y_i f_0 q = y_i f_1 q$ or $u_i q = v_i q$, meaning that the relation $u_i = v_i$ holds in Q. Furthermore, if the relations $u_i = v_i$ hold in a group P generated by $\{x_1, \dots, x_n\}$, then there is a homomorphism $p \colon F \to P$ with $f_0 p = f_1 p$. The group homomorphism $\varinjlim p \colon Q \to P$ with $q(\varinjlim p) = p$ then shows that P is a quotient of Q, so

Q is the largest group satisfying the relations given in the presentation. It is uniquely specified by the presentation (2.2), since coequalizers, as examples of colimits, which in turn are examples of initial objects, are uniquely specified.

2.1.3. *Regular epimorphisms.* In many concrete categories, and even in categories of algebras and homomorphisms, the underlying function of an epimorphism may not necessarily be surjective.

Example 2.1. Consider the category **Ring** of unital rings. Then the inclusion $j\colon \mathbb{Z} \hookrightarrow \mathbb{Q}$ is an epimorphism (Exercise 14), even though the underlying function is not surjective.

The concept of a regular epimorphism is intended to address this problem.

Proposition 2.2. *Consider a category* $\mathbf{C}$. *Suppose that* $q\colon X_0 \to Q$ *is the coequalizer in* $\mathbf{C}$ *of a parallel pair* $X_1 \overset{f_0}{\underset{f_1}{\rightrightarrows}} X_0$. *Then* q *is an epimorphism.*

Proof. Consider morphisms $f, g\colon Q \to P$ with $qf = qg$, say

$$qf = qg = p\,. \tag{2.3}$$

Now $f_0 p = f_0 q f = f_1 q f = f_1 p$, so by the coequalizer property, there is a unique morphism $\varinjlim p\colon Q \to P$ with $q(\varinjlim p) = p$. Thus (2.3) yields $f = \varinjlim p = g$. ☐

Definition 2.3. A *regular epimorphism* in a category $\mathbf{C}$ is a coequalizer $q\colon X_0 \to Q$ of a parallel pair $X_1 \overset{f_0}{\underset{f_1}{\rightrightarrows}} X_0$ of arrows in $\mathbf{C}$.

Regular epimorphisms are often obtained as follows.

Definition 2.4. Let $f\colon X \to Y$ be a morphism in a category $\mathbf{C}$.

(a) The *kernel pair* of f is the pullback $\ker f$ of $X \xrightarrow{f} Y \xleftarrow{f} X$.
(b) The morphism f is said to be an *effective epimorphism* if it is the coequalizer of its kernel pair $\ker f \overset{\pi_0}{\underset{\pi_1}{\rightrightarrows}} X$.

Remark 2.5. In **Set**, or concrete categories of algebras like **Gp**, **Ring**, or **Lin**, the First Isomorphism Theorem shows that each morphism f may be factorized as the product $f = em$ of an effective epimorphism e and a monomorphism m.

2.2. **Coproducts.**

2.2.1. *Disjoint unions of sets.* In the category of sets, the coproduct $\coprod_{i\in I} X_i$ is given by the disjoint union. One realization is as the union $\bigcup_{i\in I}(X_i \times \{i\})$, with insertions

$$\iota_j \colon X_j \to \bigcup_{i\in I}(X_i \times \{i\})\,; x \mapsto (x, j)$$

for $j \in I$. Of course, all the possible realizations are isomorphic.

2.2.2. *Coproducts of free algebras.* Let $\mathbf{A}$ be a category of algebras, with underlying set functor $G\colon \mathbf{A} \to \mathbf{Set}$. The forgetful functor G has a left adjoint $F\colon \mathbf{Set} \to \mathbf{A}$, assigning the free algebra XF over X to each set X. Comparing the defining properties, one sees that the free algebra $(X+Y)F$ over the disjoint union $X+Y$ of sets X and Y is the coproduct $XF * YF$ of the free algebras XF over X and YF over Y. The homomorphic insertions $\iota_X\colon XF \to (X+Y)F$ and $\iota_Y\colon YF \to (X+Y)F$ are the respective unique homomorphic extensions of the composite functions $X \to (X+Y) \xrightarrow{\eta_{X+Y}} (X+Y)FG$, $Y \to (X+Y) \xrightarrow{\eta_{X+Y}} (X+Y)FG$. Now consider homomorphisms $f\colon XF \to A$ and $g\colon YF \to A$, with restriction functions $f|_X\colon X \to AG$ and $g|_Y\colon Y \to AG$. Then the function $f|_X + g|_Y\colon X+Y \to AG$ extends to a unique coproduct homomorphism $f * g\colon (X+Y)F \to A$.

2.2.3. *Free products of groups.* Coproducts $G*H$ of groups G, H, known as *free products*, may be described in terms of presentations. Suppose that

$$G = \langle x_1, \dots, x_m \mid s_1 = t_1, \dots, s_q = t_q\rangle$$

and

$$H = \langle y_1, \dots, y_n \mid u_1 = v_1, \dots, u_r = v_r\rangle$$

are respective presentations of G and H, with disjoint generating sets $\{x_1, \dots, x_m\}$ and $\{y_1, \dots, y_n\}$. Then $G * H =$

$$\langle x_1, \dots, x_m, y_1, \dots, y_n \mid s_1 = t_1, \dots, s_q = t_q, u_1 = v_1, \dots, u_r = v_r\rangle$$

is a presentation of the free product $G * H$.

Example 2.6. Let H^2 denote the *upper half-plane* $\{x+iy \in \mathbb{C} \mid y > 0\}$. Then the *modular group* Γ is the subgroup of H^2! generated by the elements

$$S\colon z \mapsto -1/z$$

and

$$T\colon z \mapsto z+1$$

which satisfy the relations

$$S^2 = 1 \quad \text{and} \quad (ST)^3 = 1 \tag{2.4}$$

(see Exercise 17.) Indeed, there is a presentation

$$\Gamma = \langle S, T \mid S^2 = 1, (ST)^3 = 1 \rangle$$

(compare J.-P. Serre, "A Course in Arithmetic," Ch. VII) or

$$\Gamma = \langle S, U \mid S^2 = 1, U^3 = 1 \rangle$$

with $U = ST$. Thus Γ is the free product $C_2 * C_3$ of the cyclic groups

$$C_2 = \langle S \mid S^2 = 1 \rangle$$

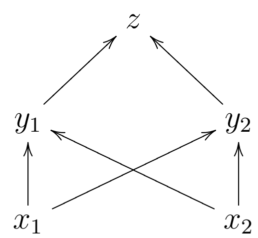
and

$$C_3 = \langle U \mid U^3 = 1 \rangle$$

(given in terms of presentations).

2.3. **Directed colimits.** Recall that a poset category J is a join semilattice if each pair of objects has a coproduct, a least upper bound. More weakly, a poset category J is said to be (*upwards*) *directed* if each pair of objects has an upper bound.

Example 2.7. The poset with Hasse diagram



is upwards directed, but does not form a join semilattice.

Definition 2.8. Let $\mathbf{C}$ be a category, and let J be a directed poset. Then the colimit $\varinjlim F$ of a functor $F\colon J \to \mathbf{C}$ is described as a *directed colimit*.

The following proposition (which is typical of analogues for more general algebras) gives an illustration of the use of directed colimits.

Proposition 2.9. *Let V be a vector space over a field K. Let J be the poset of finite-dimensional subspaces of V, ordered by containment. Let $F\colon J \to \underline{\underline{K}}$ send an inclusion $X \subseteq Y$ to the linear inclusion $X \hookrightarrow Y$. Then the directed colimit of F is V.*

Example 2.10. Consider $\mathbb{R}$ as a vector space over its subfield $\mathbb{Q}$. Then $\mathbb{R}$ is the directed colimit of its finite-dimensional $\mathbb{Q}$-subspaces.

2.4. **p-adic integers.** Let p be a prime number. For natural numbers n and r, there is a unital ring homomorphism

$$\rho_n^{n+r} \colon \mathbb{Z}/p^{n+r}\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}; x + p^{n+r}\mathbb{Z} \mapsto x + p^n\mathbb{Z}$$

given by reduction modulo p^n. Consider the functor $F \colon (\mathbb{N}, \geq) \to \mathbf{Ring}$ with $(n+r, n)F = \rho_n^{n+r}$ for natural numbers n and r.

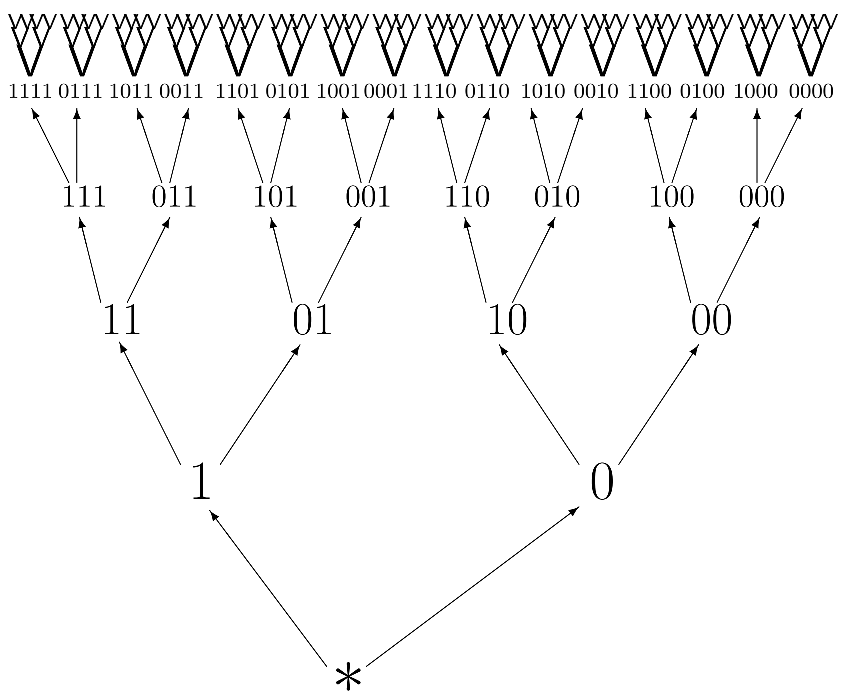
Definition 2.11. The limit $\varprojlim F$ is the ring $\mathbb{Z}_p$ of p-*adic integers*.

One may realize $\mathbb{Z}_p$ as the set of sequences

$$(x_1 + p\mathbb{Z}, x_2 + p^2\mathbb{Z}, x_3 + p^3\mathbb{Z}, \dots) \tag{2.5}$$

such that $\forall\, 0 < n \in \mathbb{N}$, $x_{n+1} + p^n\mathbb{Z} = x_n + p^n\mathbb{Z}$. (See Figure 1 below for the case $p = 2$.) The ring operations are performed componentwise. In particular, since each residue coprime to p is invertible in each quotient $\mathbb{Z}/p^n\mathbb{Z}$ for $n \in \mathbb{N}$, each p-adic number (2.5) with $x_1 \not\equiv 0 \mod p$ is invertible in $\mathbb{Z}_p$. For example, the inverse of 3 or $(1, 3, 3, 3, 3, \dots)$ in $\mathbb{Z}_2$ is $(1, 3, 3, 11, 11, \dots)$ in $\mathbb{Z}_2$. Similarly, one may solve certain algebraic equations within the rings $\mathbb{Z}_p$ (compare Exercise 21).

FIGURE 1. 2-adic integers as paths in the binary tree.



3. Exercises

(1) In the context of Definition 1.1, verify that C^D is a category.
(2) Let G be a group, realized as a category with a single object. Show that the functor category $\mathbf{Set}^G$ is essentially the category of G-sets.
(3) Let J be a category with an initial object $\perp$. For each functor $F\colon J \to \mathbf{C}$, show that $\varprojlim F = \perp F$.
(4) Show that a pullback $X \times_Z Y$ may be expressed as the equalizer of a parallel pair $X \times Y \rightrightarrows Z$ of arrows from the product $X \times Y$.
(5) Suppose that finite products exist in a category $\mathbf{C}$. Show that the equalizer of a parallel pair $X_0 \underset{f_1}{\overset{f_0}{\rightrightarrows}} X_1$ of arrows in $\mathbf{C}$ is the pullback of $X_0 \xrightarrow{1_{X_0} \times f_0} X_0 \times X_1 \xleftarrow{1_{X_0} \times f_1} X_0$.
(6) Let C be a poset category. Show that equalizer morphisms $e\colon E \to X_0$ in C are identities.
(7) Justify the claim of the final sentence of §1.1.4.
(8) Let G be a group, realized as a category. Show that a parallel pair of distinct morphisms f_0, f_1 of G has no equalizer.
(9) Let V be a finite-dimensional real vector space. Consider the endomorphism monoid $\operatorname{End} V$ as a category. Show that the equalizer e of a pair of morphisms f_0, f_1 is the insertion into V of the null space $\operatorname{Ker}(f_0 - f_1)$ of $f_0 - f_1$.
(10) Pullbacks were used to build products in slice categories $\mathbf{C}/Q$. Can you use pushouts to build coproducts in slice categories $Q/\mathbf{C}$?
(11) Show that (2.1) is the coequalizer of the parallel pair $B \underset{\pi_1}{\overset{\pi_0}{\rightrightarrows}} X$ in §2.1.1.
(12) Let $f\colon A \to B$ be a homomorphism of abelian groups. Show that the cokernel $B \to \operatorname{Coker} f$ is the coequalizer of the parallel pair $A \underset{0}{\overset{f}{\rightrightarrows}} B$.

(13) Show that the coequalizer property of presentations discussed in §2.1.2 may be used to conclude that the symmetric group

$$S_n = \langle t_1, \dots, t_{n-1} \mid t_1^2 = \dots = t_{n-1}^2 = 1, \quad t_r t_{r+1} t_r = t_{r+1} t_r t_{r+1} \text{ for } 0 < r < n-1, \quad t_r t_s = t_s t_r \text{ for } |r-s| > 1 \text{ and } 0 < r, s < n \rangle$$

of positive degree n is a quotient of the braid group

$$B_n = \langle t_1, \dots, t_{n-1} \mid t_r t_{r+1} t_r = t_{r+1} t_r t_{r+1} \text{ for } 0 < r < n-1, \quad t_r t_s = t_s t_r \text{ for } |r-s| > 1 \text{ and } 0 < r, s < n \rangle$$

of positive degree n.

(14) Show that the inclusion $j\colon \mathbb{Z} \hookrightarrow \mathbb{Q}$ is an epimorphism in the category of unital rings and homomorphisms.

(15) Suppose that $e\colon E \to X_0$ is the equalizer of a pair $X_0 \underset{f_1}{\overset{f_0}{\rightrightarrows}} X_1$ of parallel arrows. Show that e is a monomorphism.

(16) Let $f\colon X \to Y$ be a function. Describe the kernel pair

$$\ker f \underset{\pi_1}{\overset{\pi_0}{\rightrightarrows}} X$$

of f in the category **Set**.

(17) In the modular group Γ, show that S and T satisfy the relations (2.4).

(18) Show that the modular group Γ is a quotient of the braid group B_3. [Hint: Consider the generators $t_1t_2t_1$ and t_1t_2 of B_3.]

(19) Verify Proposition 2.9.

(20) Show that every group is obtained as the directed colimit of its finitely generated subgroups.

(21) Show that -1 has a square root in the ring $\mathbb{Z}_5$ of 5-adic integers.

(22) Can you justify the computation $-1 = 1 + 2 + 2^2 + 2^3 + \dots$ in the ring $\mathbb{Z}_2$ of 2-adic integers?