CONTINUITY

Abstract. Continuity, tensor products, complete lattices, the Tarski Fixed Point Theorem, existence of adjoints, Freyd's Adjoint Functor Theorem

1. Continuity

1.1. **Preserving limits and colimits.** Suppose that $F\colon J \to C$ and $R\colon C \to D$ are functors. Consider the limit diagrams

$$\varprojlim F \xrightarrow{\pi_j} jF \tag{1.1}$$

for F in C and

$$\varprojlim (FR) \xrightarrow{\varpi_j} jFR \tag{1.2}$$

for FR in D, as well as the colimit diagrams

$$jF \xrightarrow{\iota_j} \varinjlim F \tag{1.3}$$

for F in C and

$$jFR \xrightarrow{i_j} \varinjlim (FR) \tag{1.4}$$

for FR in D.

Definition 1.1. (a) Suppose that F has a limit (1.1). The functor R *preserves* the limit of F if $(\varprojlim F)R \xrightarrow{\pi_j R} jFR$ is a limit of FR in D.

(b) Suppose that F has a colimit (1.3). Then the functor R *preserves* the colimit of F if $jFR \xrightarrow{\iota_j R} (\varinjlim F)R$ is a colimit of FR in D.

Remark 1.2. The preservation property expressed in Definition 1.1(a) is often summarized by the equation

$$\varprojlim (FR) = (\varprojlim F)R\,.$$

Dually,

$$\varinjlim (FR) = (\varinjlim F)R$$

summarizes Definition 1.1(b).

Example 1.3. Let $U\colon \mathbf{Gp} \to \mathbf{Set}$ be the usual forgetful functor from groups to sets.

(a) If $G_1 \times G_2$ is a product of groups, then $(G_1 \times G_2)U = G_1U \times G_2U$. Thus U preserves the product of two groups.

(b) Recall that the modular group Γ is the coproduct $C_2 * C_3$ of the respective cyclic groups of orders 2 and 3. (See Example 2.6, Limits.) However, ΓU is an infinite set, so it is not the disjoint union of the finite sets C_2U and C_3U. Thus U does not preserve the coproduct $C_2 * C_3$.

Example 1.4. Let $G\colon \mathbf{Set} \to \mathbf{Gp}$ be the free group functor.

(a) Suppose that $X_1 + X_2$ is the disjoint union of sets X_1 and X_2. Then $(X_1 + X_2)G = X_1G * X_2G$, so G preserves the coproduct of two sets.

(b) Let X be a singleton set. Then $XG \cong (X \times X)G \cong \mathbb{Z}$, but $(X \times X)G \cong \mathbb{Z} \not\cong \mathbb{Z} \times \mathbb{Z} \cong XG \times XG$, so G does not preserve the product $X \times X$.

1.1.1. *Exactness and continuity.* Let J and C be categories. Limits of functors $F\colon J \to C$ in C are classified according to the nature of the category J. Specifically, $\varprojlim F$ is:

- *small* if J is small;
- *finite* if J is finite;
- *directed* if J is directed.

A similar classification applies to colimits. (Compare Definition 2.8, Limits.) This classification extends to categories and functors.

Definition 1.5. Let C be a category.

(a) C is *complete* if all small limits exist in C.
(b) C is *cocomplete* if all small colimits exist in C.
(c) C is *left exact* if all finite limits exist in C.
(d) C is *right exact* if all finite colimits exist in C.

Definition 1.6. Let $R\colon C \to D$ be a functor.

(a) R is *continuous* if it preserves all small limits in C.
(b) R is *cocontinuous* if it preserves all small colimits in C.
(c) R is *left exact* if it preserves all finite limits in C.
(d) R is *right exact* if it preserves all finite colimits in C [1].

Example 1.7 (Distributive lattices)**.** Let $(L, \vee, \wedge, 0, 1)$ be a bounded lattice. For an element x of L, consider the multiplication

$$S(x)\colon L \to L; y \mapsto x \wedge y\,.$$

[1] See Exercise 10 for a justification of the terminology of Definition 1.6(d).

Note that $S(x)\colon L \to L$ is a functor. Since the only nontrivial colimits in a poset category are coproducts, the multiplications $S(x)$ for $x \in L$ are right exact if and only if

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

for $x, y, z \in L$, i.e., if and only if the lattice is *distributive.*

1.2. **Continuity and adjunction.**

1.2.1. *Limits as adjoints.*

Proposition 1.8. *Let J and C be categories. Suppose that for each functor $F\colon J \to C$, the limit $\varprojlim F$ exists in C. Then the functor*

$$[\quad]\colon C \to C^J; X \mapsto [X]$$

is left adjoint to the functor

$$\varprojlim\colon C^J \to C; F \mapsto \varprojlim F\,.$$

Proof. For the adjunction isomorphism

$$C^J([Y], F) \cong C(Y, \varprojlim F)\,, \tag{1.5}$$

note that a natural transformation $f\colon [Y] \to F$ with components $f_j\colon Y \to jF$ corresponds to a unique C-morphism $\varprojlim f\colon Y \to \varprojlim F$. Conversely, a C-morphism $l\colon Y \to \varprojlim F$ determines a unique natural transformation $f\colon [Y] \to F$ with components $f_j = l\pi_j$ for j in J, such that $l = \varprojlim f$. □

Corollary 1.9. *Let J and C be categories, with a functor $F\colon J \to C$. The isomorphism* (1.5) *holds for each object Y of C if and only if the functor F has a limit.*

Proof. The existence of the limit is equivalent to the existence of a unique C-morphism $\varprojlim f\colon Y \to \varprojlim F$ for each natural transformation $f\colon [Y] \to F$. □

1.2.2. *Continuity of right adjoints.*

Proposition 1.10. *Suppose that a functor $R\colon C \to D$ has a left adjoint $S\colon D \to C$. Then R is continuous.*

Proof. Suppose that a functor $F\colon J \to C$ with small domain has a limit $\varprojlim F$ in C. For each object X of C, an isomorphism

$$C^J\big([XS], F\big) \cong D^J\big([X], FR\big) \tag{1.6}$$

is provided by the adjunction isomorphism

$$C(XS, jF) \cong D(X, jFR)$$

for each object j of J, sending the component h_j of an element of the left hand side of (1.6) to the corresponding component of a natural transformation from the right hand side. Now consider the string of isomorphisms

$$\begin{aligned} D^J\big([X], FR\big) &\cong C^J\big([XS], F\big) \\ &\cong C(XS, \underleftarrow{\lim} F) \cong D\big(X, (\underleftarrow{\lim} F)R\big) \end{aligned} \tag{1.7}$$

coming respectively from (1.6), (1.5), and the adjunction. Considering the isomorphism between the extremes of (1.7), Corollary 1.9 then shows that $\underleftarrow{\lim}\,(FR) = (\underleftarrow{\lim}\, F)R$, as required. □

Corollary 1.11. *Suppose that a functor $S\colon D \to C$ has a right adjoint $R\colon C \to D$. Then S is cocontinuous.*

1.3. **Tensor products of modules.** Let S be a unital, commutative ring. Let $\underline{\underline{S}}$ be the category of unital S-modules. Recall the adjunction

$$\underline{\underline{S}}(Z \otimes Y, X) \cong \underline{\underline{S}}(Z, \underline{\underline{S}}(Y, X))$$

that defines the tensor product $Z \otimes Y$ of S-modules Z and Y, with left adjoint $F_Y\colon \underline{\underline{S}} \to \underline{\underline{S}}; Z \mapsto Z \otimes Y$ (compare Proposition 2.5, Functors and adjunctions), as well as the isomorphism

$$\underline{\underline{S}}(Z \otimes Y, X) \cong \underline{\underline{S}}(Z, Y; X) \tag{1.8}$$

with the set of bilinear functions from $Z \times Y$ to an S-module X. Setting $X = Z \otimes Y$ in (1.8) yields a bilinear function

$$\otimes\colon Z \times Y \to Z \otimes Y; (z, y) \mapsto z \otimes y \tag{1.9}$$

corresponding to $1_{Z \otimes Y}$ on the left. Elements of the image of (1.9) inside $Z \otimes Y$ are described as *primitive*. The primitive elements may form a proper subset of $Z \otimes Y$ (Exercise 7).

The isomorphism $Z \times Y \cong Y \times Z$ yields the commutative law

$$Z \otimes Y \cong Y \otimes Z$$

for tensor products. The cocontinuity of F_Y given by Corollary 1.11 implies the distributive law

$$Y \otimes \sum_{i \in I} Z_i \cong \sum_{i \in I} Y \otimes Z_i \tag{1.10}$$

for modules Y and Z_i with i in an arbitrary index set I.

Proposition 1.12. *There is a natural transformation* $\lambda\colon F_S \to 1_{\underline{\underline{S}}}$ *with isomorphic components* $\lambda_Z\colon Z\otimes S \to Z; z\otimes s \mapsto zs$.

Proof. For S-modules Z and X, one has isomorphisms

$$\underline{\underline{S}}(Z\otimes S, X) \cong \underline{\underline{S}}(Z, \underline{\underline{S}}(S,X)) \cong \underline{\underline{S}}(Z,X)\,, \tag{1.11}$$

where the latter isomorphism follows from the two mutually inverse isomorphisms

$$X \to \underline{\underline{S}}(S,X); x \mapsto (S\to X; 1\mapsto x)$$

and

$$\underline{\underline{S}}(S,X) \to X; f \mapsto 1f$$

given by the freeness of the S-module S on $\{1\}$. Comparing the extreme ends of (1.11), the isomorphism $Z\otimes S\cong Z$ is given by the uniqueness of adjoints. □

Proposition 1.13. *Tensor products of free modules are free.*

Proof. Recall that the free S-module on a set I is the coproduct $\sum_{i\in I} S$ of copies of the free module S indexed by I. For sets I and J, the isomorphisms

$$\sum_{i\in I} S \otimes \sum_{j\in J} S \cong \sum_{(i,j)\in I\times J} S\otimes S \cong \sum_{(i,j)\in I\times J} S$$

are given by the distributive law (1.10) and Proposition 1.12. □

Proposition 1.14. *Let* X *be an* S*-module. Let* K *be an ideal of* S.

(a) *There are mutually inverse isomorphisms*

$$X\otimes (S/K) \to X/(XK); x\otimes (s+K)\mapsto xs + XK$$

and

$$X/(XK) \to X\otimes (S/K); x + XK \mapsto x\otimes 1\,.$$

(b) *In particular,*

$$S/K\otimes S/L \cong S/(K+L)$$

with $K+L = \{k+l \mid k\in K, l\in L\}$ *for a second ideal* L *of* S.

Proposition 1.14 is especially useful for computing tensor products of abelian groups. For instance,

$$\mathbb{Z}/_m \otimes \mathbb{Z}/_n \cong \mathbb{Z}/_d \tag{1.12}$$

with positive integers m and n, if $\gcd\{m,n\} = d$ (Exercise 9).

2. Existence of adjoints

2.1. **Complete lattices.**

Proposition 2.1. *Let $(X, \leq)$ be a poset, construed as a category. Then the following conditions are equivalent:*

(a) *For each function $A\colon J \to X; j \mapsto x_j$, the greatest lower bound*
$$\prod\{x_j \mid j \in J\}$$
exists;

(b) *For each function $A\colon J \to X; j \mapsto x_j$, the least upper bound*
$$\sum\{x_j \mid j \in J\}$$
exists.

Proof. Suppose that (a) holds. Let $A\colon J \to X; j \mapsto x_j$ be a function. Define
$$U = \{u \in X \mid \forall\ j \in J\ ,\ x_j \leq u\}\ ,$$
the set of upper bounds of the image JA. Let $I\colon U \hookrightarrow X; u \mapsto u$ be the inclusion function. By (a), the product $\prod U$ exists. Consider $x_j \in JA$. Now
$$\forall\ u \in U\ ,\ x_j \leq u \quad \Rightarrow \quad x_j \leq \prod U\ ,$$
so $\prod U$ is an upper bound for JA. Suppose that u is an upper bound for JA. On the other hand, if u is an upper bound for JA, one has $u \in U$, and so $\prod U \leq u$. Thus $\prod U$ is the least upper bound for JA. The implication (b) $\Rightarrow$ (a) is dual to (a) $\Rightarrow$ (b). □

Definition 2.2. A poset that satisfies the equivalent conditions of Proposition 2.1 is called a *complete lattice*.

2.2. **Tarski's Fixed Point Theorem.**

Theorem 2.3. *Let $(X, \leq)$ be a complete lattice. Let $T\colon X \to X$ be a functor (order-preserving map). Consider the set*
$$X_T = \{x \in X \mid xT = x\}$$
of fixed points of T. Then the induced poset $(X_T, \leq)$ is a complete lattice.

Proof. Let A be a subset of X_T. Define
$$B = \{b \in X \mid \forall\ a \in A\ ,\ a \leq bT \leq b\}\ .$$

Let g be the greatest lower bound $\prod B$ of B in X. Then for $b \in B$, one has $g \leq b$, and so $gT \leq bT \leq b$. Thus gT is a lower bound for B, and therefore $gT \leq g$.

Next, let l be the least upper bound $\sum A$ of A in X. Then

$$(\forall\ a \in A\,,\ \forall\ b \in B\,,\ a \leq bT) \Rightarrow (\forall\ b \in B\,,\ l \leq bT \leq b) \Rightarrow l \leq g\,.$$

Thus g is an upper bound for A in X, i.e.

$$\forall\ a \in A\,,\ a \leq g\,.$$

Applying T yields

$$\forall\ a \in A\,,\ aT = a \leq gT \leq g\,,$$

so $g \in B$. Applying T again yields

$$\forall\ a \in A\,,\ aT = a \leq gT^2 \leq gT\,,$$

so $gT \in B$. Since $g = \prod B$, one has $g \leq gT$. Thus $gT = g \in X_T$. This means that g, which was an upper bound for A in X, is actually an upper bound for A in X_T.

If u is any upper bound for A in X_T, then

$$(\forall\ a \in A\,,\ a \leq uT \leq u) \Rightarrow u \in B \Rightarrow g \leq u\,.$$

Thus g, the greatest lower bound for B in X, is the least upper bound for A in X_T. □

Corollary 2.4. *Consider functions $f\colon A \to B$ and $g\colon B \to A$.*

(a) **Banach Decomposition Theorem.** *There are disjoint union decompositions $A = A_1 + A_2$ and $B = B_1 + B_2$ with $A_1 f = B_1$ and $B_2 g = A_2$.*

(b) **Cantor-Schröder-Bernstein Theorem.** *If f and g inject, then A and B are isomorphic.*

Proof. (a) Consider the following four functors:

$$S_1\colon (2^A, \subseteq) \to (2^B, \subseteq); Y \mapsto Yf\,;$$
$$S_2\colon (2^B, \subseteq) \to (2^B, \supseteq); Z \mapsto B \smallsetminus Z\,;$$
$$S_3\colon (2^B, \supseteq) \to (2^A, \supseteq); Z \mapsto Zg\,;$$
$$S_4\colon (2^A, \supseteq) \to (2^A, \subseteq); Y \mapsto A \smallsetminus Y\,.$$

Their composite $T\colon (2^A, \subseteq) \to (2^A, \subseteq)$ is a functor on the complete lattice $(2^A, \subseteq)$, and thus has a fixed point A_1, i.e.,

$$A_1 = A_1 T = A_1 S_1 S_2 S_3 S_4\,.$$

Now set $A_2 = A_1 S_1 S_2 S_3$, $B_2 = A_1 S_1 S_2$, and $B_1 = A_1 S_1$.

(b) The disjoint union $(f\colon A_1 \to B_1) + (g^{-1}\colon A_2 \to B_2)$ is a bijection from A to B. □

2.3. **Existence of adjoints on posets.** Proposition 1.10 showed that if a functor R is a right adjoint, then it is continuous. The following result gives the converse for poset categories, if the domain of R is a complete lattice.

Proposition 2.5. *Suppose that $R\colon A \to B$ is a continuous functor from a complete lattice to a poset. Then R has a left adjoint $S\colon B \to A$.*

Proof. Define

$$S\colon B \to A; x \mapsto \prod\{z \in A \mid x \leq z^R\}.$$

Note that for $x_1, x_2 \in A$, one has

$$\begin{aligned} x_1 \leq x_2 &\Rightarrow \{z \in A \mid x_1 \leq z^R\} \supseteq \{z \in A \mid x_2 \leq z^R\} \\ &\Rightarrow \prod\{z \in A \mid x_1 \leq z^R\} \leq \prod\{z \in A \mid x_2 \leq z^R\} \Rightarrow x_1^S \leq x_2^S, \end{aligned}$$

so S is a functor. Then for the adjointness, one has the unit relationship

$$x \leq \prod\{z^R \mid z \in A,\ x \leq z^R\} = \Big(\prod\{z \mid z \in A,\ x \leq z^R\}\Big)^R = x^{SR}$$

for x in B, the first equality holding by the continuity of R. Dually,

$$y^{RS} = \prod\{z \in A \mid y^R \leq z^R\} \leq y$$

for y in A. □

Dually, one has the following:

Corollary 2.6. *Suppose that $S\colon B \to A$ is a cocontinuous functor from a complete lattice to a poset. Then S has a right adjoint $R\colon A \to B$.*

2.4. **Complete Heyting algebras.** Let $(L, \vee, \wedge, 0, 1)$ be a bounded lattice. Recall that L is distributive if and only if the functors

$$S(x)\colon L \to L; y \mapsto x \wedge y$$

are right exact for all x in L (Example 1.7). If L is a complete lattice, and the functors $S(x)$ are cocontinuous for all x in L, i.e.

$$\forall\ J \subseteq L,\ x \wedge \sum\{y \mid y \in J\} = \sum\{x \wedge y \mid y \in J\} \tag{2.1}$$

for all x in L, then L is called a *completely distributive lattice*. However, Corollary 2.6 shows that each $S(y)$ has a right adjoint

$$R(y)\colon L \to L; z \mapsto y\backslash z$$

namely with

$$L\big(xS(y), z\big) \cong L\big(x, zR(y)\big)$$

or

$$x \wedge y \leq z \quad \Leftrightarrow \quad x \leq y\backslash z\,,$$

so that $(L, \vee, \wedge, \backslash, 0, 1)$ forms a *complete Heyting algebra*.

Example 2.7. Let X be a set. With the Boolean implication

$$U \to V = (X \smallsetminus U) \cup V ,$$

the power set 2^X forms a complete Heyting algebra $(2^X, \cup, \cap, \to, \emptyset, X)$. (Compare Example 2.22, Categories.)

2.4.1. *Topology.* Let X be a set. Let T be a complete sublattice of 2^X for which the inclusion $T \hookrightarrow 2^X$ is a cocontinuous bounded lattice homomorphism. Then T is a complete Heyting algebra. Indeed, the complete distributive law (2.1) in T is inherited from 2^X. Such a sublattice T of 2^X is called a *topology* on the set X, and the pair (X, T) is a *topological space*. Elements of T are called *open subsets of X in the topology T*.

Let (X, T) and (Y, U) be topological spaces Consider a function $f \colon X \to Y$, and the right adjoint

$$f^{-1} \colon (2^Y, \subseteq) \to (2^X, \subseteq); Z \mapsto f^{-1}Z \tag{2.2}$$

to the functor

$$(2^X, \subseteq) \to (2^Y, \subseteq); W \mapsto Wf \tag{2.3}$$

(compare Exercise 13). The function f is said to be *continuous* if f^{-1} restricts to a functor $f^{-1} \colon (U, \subseteq) \to (T, \subseteq)$. The large concrete category **Top** has topological spaces as its objects and continuous maps as its morphisms. It is the domain of a forgetful functor

$$U \colon \mathbf{Top} \to \mathbf{Set}$$

sending a space (X, T) to its underlying set X.

2.5. Freyd's Adjoint Functor Theorem.

Definition 2.8. Let K be a set of objects in a (possibly large) category C. Then K is said to *dominate* C if each object of C is the codomain of a C-morphism whose domain lies in K.

Note that if C has an initial object $\bot$, then $\{\bot\}$ dominates C. A poset category is dominated by the set of minimal elements of the poset. More trivially, a small category C is dominated by its object set C_0.

In order to motivate the following definition, recall that a functor $G \colon C \to D$ has a left adjoint if for each object X of D, the comma category $([X] \downarrow G)$ has an initial object. (Compare Theorem 1.14, Structural specifications.)

Definition 2.9. A functor $G\colon C \to D$ is said to satisfy the *solution set condition* if for each object X of D, the comma category $([X] \downarrow G)$ possesses a dominating set.

Freyd's Adjoint Functor Theorem gives a very powerful converse to Proposition 1.10, and a broad generalization of Proposition 2.5:

Theorem 2.10. *Suppose that C is a complete, locally small category. If a continuous functor $G\colon C \to D$ satisfies the solution set condition, then it has a left adjoint.*

See Section III.3.5 in Post-Modern Algebra for the proof of Freyd's Adjoint Functor Theorem.

3. Exercises

(1) Let U and V be finite-dimensional real vector spaces. Specify a functor $S\colon \mathbf{Set} \to \mathbf{Lin}$ such that the equation

$$\dim(U \oplus V) = \dim U + \dim V$$

may be interpreted to express the fact that S preserves certain coproducts.

(2) Let $*\colon \mathbf{Mon} \to \mathbf{Gp}$ be the functor sending a monoid to its group of units. Show that $*$ preserves products.

(3) Let $\mathcal{P}^{<\infty}\mathbb{N}$ be the set of subsets of $\mathbb{N}$ whose complement is finite. Consider $(\mathcal{P}^{<\infty}\mathbb{N}, \subseteq)$ as a poset category.
   (a) Show that finite products exist in $\mathcal{P}^{<\infty}\mathbb{N}$.
   (b) Show that infinite products may not exist in $\mathcal{P}^{<\infty}\mathbb{N}$.

(4) In the adjunction of Proposition 1.8, determine the unit and counit.

(5) Formulate the duals of Proposition 1.8 and Corollary 1.9.

(6) Suppose that X_i, for $1 \le i \le 3$, are unital modules over a unital, commutative ring S. Verify the associative law

$$(X_1 \otimes X_2) \otimes X_3 \cong X_1 \otimes (X_2 \otimes X_3)$$

for tensor products.

(7) Let K be a finite field of order q. Consider the row vector spaces $Z = K_1^2$ and $Y = K_1^3$.
   (a) Show that $|Z \times Y| = q^5$.
   (b) Show that $|Z \otimes Y| = q^6$.
   (c) Conclude that the tensor product $Z \otimes Y$ contains elements which are not primitive.

(8) Specify the inverse of the mapping λ_Z in Proposition 1.12.

(9) Derive (1.12) from Proposition 1.14.

(10) Let

$$0 \longrightarrow A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C \longrightarrow 0 \tag{3.1}$$

be an exact sequence of abelian groups. Let Y be an abelian group. Show that

$$A \otimes Y \xrightarrow{fF_Y} B \otimes Y \xrightarrow{gF_Y} C \otimes Y \longrightarrow 0 \tag{3.2}$$

is an exact sequence of abelian groups.

[According to the terminology of Definition 1.6(d), the functor F_Y, as a left adjoint, is right exact. The exactness in (3.2) of the image of the right hand end of the original sequence (3.1) is the motivation for this terminology.]

(11) Let S be a unital, commutative ring.
   (a) For positive integers m and n, prove that
$$S_m^m \otimes S_n^n \cong S_{mn}^{mn} .$$
   (b) For matrices $[x_{ij}]$ in S_m^m and $[y_{kl}]$ in S_n^n, prove that the element $[x_{ij}] \otimes [y_{kl}]$ of $S_m^m \otimes S_n^n$ corresponds to $[x_{ij}y_{kl}]$ in S_{mn}^{mn} under the isomorphism of (a).

(12) Perform the Banach decomposition for the pair of functions $f\colon \mathbb{Z} \to \mathbb{Z}; n \mapsto 2n$ and $g\colon \mathbb{Z} \to \mathbb{Z}; n \mapsto 3n$.

(13) Show that (2.2) is a functor that is right adjoint to (2.3).

(14) Let X be a set.
   (a) Show that $\{\emptyset, X\}$ forms a topology (the *indiscrete topology* on X).
   (b) Show 2^X forms a topology (the *discrete topology* on X).
   (c) Show that $U\colon \mathbf{Top} \to \mathbf{Set}$ is left adjoint to an *indiscrete topology functor* which sends a set X to the space X with the indiscrete topology.
   (d) Show that $U\colon \mathbf{Top} \to \mathbf{Set}$ is right adjoint to a *discrete topology functor* which sends a set X to the space X with the discrete topology.
   (e) Conclude that the forgetful functor $U\colon \mathbf{Top} \to \mathbf{Set}$ is both continuous and cocontinuous.

(15) Let Y be a subset of a topological space (X, T). Show that
$$U = \{Z \subseteq Y \mid \exists\, G \in T \,.\, Z = G \cap Y\}$$
is a topology on Y (the *subspace topology* on Y).

(16) Let $(X, \leq)$ be a poset. Show that the set $x^{\geq} = \{a \in X \mid x \geq a\}$ of downsets (for $x \in X$) forms a topology on X (the *Alexandrov topology*). (Compare Example 1.7, Structural specifications.)

(17) Let X be a set. Let $(B, \cap, X)$ be a submonoid of $(2^X, \cap, X)$. Show that $\{Y \subseteq X \mid \forall\, y \in Y\,,\ \exists\, Z \in B\,.\, y \in Z \subset Y\}$ is a topology on X (the *topology spanned* or *generated by the base* B). [Let S be a subset of 2^X. Let $(\langle S\rangle, \cap, X)$ be the submonoid of $(2^X, \cap, X)$ generated by S. Then the topology spanned by the base $\langle S\rangle$ is called the *topology spanned by the subbase* S.]

(18) Let $G\colon C \to D$ be a functor with small domain and codomain.
   (a) Show that R satisfies the solution set condition.
   (b) Show that Proposition 2.5 is a special case of Theorem 2.10.