CATEGORIES

ABSTRACT. Categories, monoids and pre-ordered sets. Products, sums, exponents, Currying. Lattices and Heyting algebras.

1. CATEGORIES

1.1. Definition and examples. While monoids abstract monoids of functions, categories abstract more general systems of functions.

Definition 1.1. A *category* $\mathbf{C} = (\mathbf{C}_0, \mathbf{C}_1)$ consists of two classes, a class $\mathbf{C}_0$ of *objects* and a class $\mathbf{C}_1$ of *morphisms* or *arrows*, together with three functions:

(a) The *identity function*

$$\epsilon\colon \mathbf{C}_0 \to \mathbf{C}_1; X \mapsto 1_X\,;$$

(b) the *tail function* or *domain function*

$$\partial_0\colon \mathbf{C}_1 \to \mathbf{C}_0; f \mapsto f^{\partial_0}\,, \text{ and}$$

(c) the *head function* or *codomain function*

$$\partial_1\colon \mathbf{C}_1 \to \mathbf{C}_0; f \mapsto f^{\partial_1}\,,$$

such that for each object X, one has $1_X\partial_0 = X = 1_X\partial_1$.

Pictorially,

$$f^{\partial_0} \xrightarrow{\ f\ } f^{\partial_1} \text{ or } f\colon f^{\partial_0} \to f^{\partial_1}$$

or verbally: f *goes* or *leads* from f^{∂_0} to f^{∂_1}.

For objects X and Y, define $\mathbf{C}(X,Y)$ as the class of all arrows from X to Y. Then for objects U, V, W, X, there is a *composition function*

$$\mathbf{C}(U,V) \times \mathbf{C}(V,W) \to \mathbf{C}(U,W); (f,g) \mapsto fg$$

such that

$$1_U f = f = f 1_V$$

for f in $\mathbf{C}(U,V)$ and the *associative law*

$$(fg)h = f(gh) \tag{1.1}$$

holds for f in $\mathbf{C}(U,V)$, g in $\mathbf{C}(V,W)$, and h in $\mathbf{C}(W,X)$ — Figure 1.

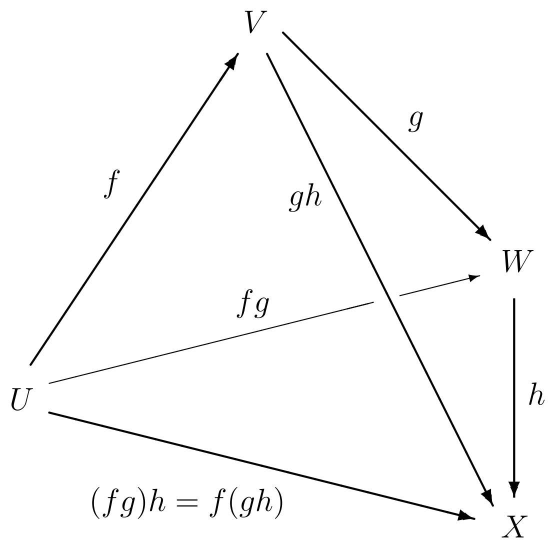


FIGURE 1. The associative law (1.1)

Example 1.2. The *category* **Set** *of sets* has the (proper) class of all sets as its object class $\mathbf{Set}_0$. For sets X and Y, one has $\mathbf{Set}(X, Y) = Y^X$, the set of all functions from X to Y. Identities and composition as usual.

Example 1.3. The category **Lin** has the class of all real vector spaces as its object class. For real vector spaces U and V, one has $\mathbf{Lin}(U, V)$ as the set of all linear transformations from U to V. Identities and composition as usual.

1.2. **Small categories.** A category $\mathbf{C}$ is *small* if its object class $\mathbf{C}_0$ is a set. The category of sets is *large*, i.e., not small. However, a category $\mathbf{C}$ is *locally small* if for each pair X, Y of objects of $\mathbf{C}$, the class $\mathbf{C}(X, Y)$ is a set. Thus **Set** is locally small.

Example 1.4. Let $(M, \cdot, 1)$ be a monoid. A category $\underline{\underline{M}}$ is defined by taking $\underline{\underline{M}}_0$ as a singleton $\{X\}$, and $\underline{\underline{M}}_1$ as the set M, with $1_X = 1$ and composition of arrows defined by the associative multiplication in the monoid M. Then $\underline{\underline{M}}$ is a small category (with one object and many morphisms).

Definition 1.5. Let V be a relation on a set X.

(a) V is *reflexive* if $\forall\ x \in X\ ,\ x\ V\ x$.
(b) The *converse* V' of V is defined by $(x, y) \in V' \ \Leftrightarrow\ (y, x) \in V$.
(c) V is *symmetric* if $V = V'$.
(d) V is *antisymmetric* if $\forall\ (x, y) \in V\ ,\ (y, x) \in V \Rightarrow x = y$.

(e) V is *transitive* if $\forall\ (x, y), (y, z) \in V\ ,\ (x, z) \in V$.
(f) V is an *equivalence relation* if it is reflexive, symmetric and transitive.
(g) (X, V) or V is a *pre-order* or a *quasi-order* if V is reflexive and transitive.
(h) V is an *order relation* if it is reflexive, antisymmetric and transitive.
(i) (X, V) is a (*partially*) *ordered set* or a *poset* if V is an order relation.

Example 1.6. For integers d, m, define the divisibility relation

$$d \mid m \quad \Leftrightarrow \quad \exists\, r \in \mathbb{Z}\,.\ m = dr\,.$$

Then $(\mathbb{Z}, |\,)$ is a pre-order, but not a poset, while $(\mathbb{N}, |\,)$ is a poset (see Exercise 3).

Proposition 1.7. *Let X be a set.*

(a) *Suppose that X is the object set of a small category $\mathbf{C}$. Then $V = \{(x, y) \in X^2 \mid \mathbf{C}(x, y) \neq \emptyset\}$ is a pre-order on X.*
(b) *Let (X, V) be a preorder. Then the functions $\epsilon\colon x \mapsto (x, x)$, $\partial_0\colon (x, y) \mapsto x$, and $\partial_1\colon (x, y) \mapsto y$ make (X, V) a category with a uniquely defined composition.*

Definition 1.8 (Equivalence classes)**.** Let E be an equivalence relation on a set X.

(a) For each element x of X, the *equivalence class* x^E is

$$\{y \in X \mid x\ E\ y\}\,.$$

(b) The *quotient set* X^E is

$$\{x^E \mid x \in X\}\,.$$

(c) The (*natural*) *projection* is

$$\operatorname{nat} E\colon X \to X^E; x \mapsto x^E\,.$$

Proposition 1.9. *Let (X, V) be a preorder.*

(a) *The relation E on X defined by*

$$x\ E\ y \quad \Leftrightarrow \quad x\ V\ y \text{ and } y\ V\ x$$

is an equivalence relation on X.

(b) *The relation*

$$x^E \leq y^E \quad \Leftrightarrow \quad x\ V\ y$$

is a well-defined order relation on the quotient set X^E.

1.2.1. *Concrete and abstract categories.* A category $\mathbf{C}$ is *concrete* if its objects are sets, its morphisms are functions, and the composition in $\mathbf{C}$ is just composition of functions. Thus **Set** and **Lin** are (large) concrete categories.

If M is a monoid of functions on a set X, then the category $\underline{\underline{M}}$ of Example 1.4 is a small concrete category. On the other hand, the category that is constructed from the divisibility preorder $(\mathbb{Z}, |\,)$ by Proposition 1.7(b) is not concrete.

General categories may be described as *abstract*, to signify that they are not (assumed to be) concrete.

1.3. **Dual categories.** Categories facilitate a precise formulation of duality.

Definition 1.10. Suppose that $\mathbf{C} = (\mathbf{C}_0, \mathbf{C}_1)$ is a category, equipped with identity function ϵ, tail function ∂_0, and head function ∂_1. Then the *dual* or *opposite category* $\mathbf{C}^{\mathsf{op}}$ has object class $\mathbf{C}_0$, morphism class $\mathbf{C}_1$, identity function ϵ, tail function ∂_1, and head function ∂_0. In particular, $\mathbf{C}^{\mathsf{op}}(X, Y) = \mathbf{C}(Y, X)$ for objects X, Y of $\mathbf{C}$ or $\mathbf{C}^{\mathsf{op}}$: the arrows $X \xrightarrow{f} Y$ of $\mathbf{C}^{\mathsf{op}}$ are reversed from arrows $X \xleftarrow{f} Y$ of $\mathbf{C}$.

Consider objects U, V, W of $\mathbf{C}^{\mathsf{op}}$. The composition function

$$\mathbf{C}^{\mathsf{op}}(U, V) \times \mathbf{C}^{\mathsf{op}}(V, W) \to \mathbf{C}^{\mathsf{op}}(U, W); (f, g) \mapsto f \circ g$$

within $\mathbf{C}^{\mathsf{op}}$ is exactly the same function as the composition function

$$\mathbf{C}(V, U) \times \mathbf{C}(W, V) \to \mathbf{C}(W, U); (f, g) \mapsto gf$$

within $\mathbf{C}$.

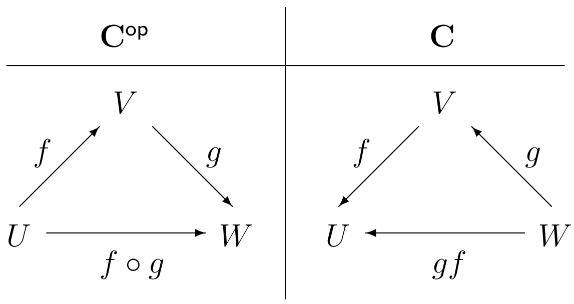


FIGURE 2. Duality of composition

Example 1.11. Let (X, V) be a pre-order category. Then the dual category $(X, V)^{\mathsf{op}}$ is the category determined by the converse preorder (X, V').

1.4. **Special morphisms.** Generalizing from **Set**:

Definition 1.12. Let $f\colon X \to Y$ be a morphism in a category $\mathbf{C}$.

(a) The morphism f is a *monomorphism* if:

$$\forall\ W\ ,\ \forall\ g_1\colon W \to X, g_2\colon W \to X\ ,\ g_1 f = g_2 f \Rightarrow g_1 = g_2\ .$$

(b) The morphism f is an *epimorphism* if:

$$\forall\ Z\ ,\ \forall\ h_1\colon Y \to Z, h_2\colon Y \to Z\ ,\ f h_1 = f h_2 \Rightarrow h_1 = h_2\ .$$

(c) A morphism $r\colon Y \to X$ is a *right inverse* for f if

$$fr = 1_X\ . \tag{1.2}$$

(d) The morphism f is *right invertible* if it has a right inverse.

(e) A morphism $s\colon Y \to X$ is a *left inverse* for f if

$$sf = 1_Y\ . \tag{1.3}$$

(f) The morphism f is *left invertible* if it has a left inverse.

(h) The morphism g is said to be *invertible* or an *isomorphism* if it is both right and left invertible.

Duality: If f is a monomorphism in $\mathbf{C}$, it is an epimorphism in $\mathbf{C}^{\mathsf{op}}$, and so on.

Proposition 1.13. *Suppose that $f\colon x \to y$ is a morphism in a poset category (X, V).*

(a) *f is a monomorphism and an epimorphism.*

(b) *If $x \neq y$, the morphism f is neither right nor left invertible.*

Proposition 1.14. *Let $f\colon X \to Y$ be a morphism in a category $\mathbf{C}$.*

(a) *If f is right invertible, it is a monomorphism.*

(b) *If f is left invertible, it is an epimorphism.*

Proof. (a) If f has right inverse r, and $g_1 f = g_2 f$, then

$$g_1 = g_1 1_X = g_1 f r = g_2 f r = g_2 1_X = g_2\ .$$

The statement (b) is dual to (a). □

Proposition 1.15. *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms in a category $\mathbf{C}$.*

(a) *If f and g are monomorphisms, then so is fg.*

(b) *If f and g are epimorphisms, then so is fg.*

(c) *If f and g are isomorphisms, then so is fg.*

2. Products

2.1. Cartesian and tensor products. Let X_1 and X_2 be sets. The *cartesian product*

$$X_1 \times X_2 = \{(x_1, x_2) \mid x_i \in X_i\}$$

has *projections*

$$\pi_i \colon X_1 \times X_2 \to X_i; (x_1, x_2) \mapsto x_i$$

for $i = 1, 2$.

2.1.1. *Componentwise structure.* If the sets X_1 and X_2 carry algebra structure — semigroup, monoid, group, real vector space, … — then a similar structure is defined *componentwise* on $X_1 \times X_2$. If X_1 and X_2 are monoids, for example, then $X_1 \times X_2$ has the *componentwise multiplication* $(x_1, x_2) \cdot (y_1, y_2) = (x_1 \cdot y_1, x_2 \cdot y_2)$ and the *componentwise identity* $(1, 1)$.

2.1.2. *Currying.* A function with a product $X \times Y$ as domain is a function of two variables. An important process known as *Currying* in computer science, or *parametrization of a family of functions* in analysis, replaces a function of two variables with a parametrized family of functions of a single variable.

Proposition 2.1. *Let X, Y, Z be sets. Then there is an isomorphism*

$$\mathbf{Set}(X \times Y, Z) \cong \mathbf{Set}(X, \mathbf{Set}(Y, Z)) \tag{2.1}$$

taking a function

$$f \colon X \times Y \to Z; (x, y) \mapsto (x, y)f$$

to a family of functions $f_x \colon Y \to Z; y \mapsto (x, y)f$ that are parametrized by elements x of X.

2.1.3. *Tensor products of real vector spaces.* There is an analogue of Currying in linear algebra. Let U, V, W be real vector spaces. Recall that the set $\mathbf{Lin}(V, W)$ of linear transformations from V to W inherits a real vector space structure from W. In particular, there is a *zero linear transformation* $0 \colon V \to W$ with image $\{0\}$.

There is a vector space $U \otimes V$, called the *tensor product* of U and V, such that there is an isomorphism

$$\mathbf{Lin}(U \otimes V, W) \cong \mathbf{Lin}(U, \mathbf{Lin}(V, W)) \tag{2.2}$$

analogous to the Currying isomorphism (2.1) for sets. To make the analogy precise, suppose that U, V, W have respective bases X, Y, Z. Then (2.2) is a direct consequence of (2.1), taking $U \otimes V$ as the vector space with basis $X \times Y$ (Exercise 14).

2.2. **Products in categories.**

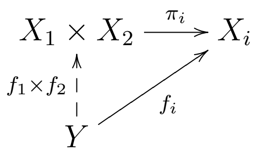
Proposition 2.2. *Let X_1 and X_2 be sets. Then for any set Y, and for any pair of functions $f_i\colon Y \to X_i$, there is a unique function*

$$f_1 \times f_2\colon Y \to X_1 \times X_2; y \mapsto (yf_1, yf_2)$$

such that $(f_1 \times f_2)\pi_i = f_i$, for $i = 1, 2$.

Here is a picture:

(2.3)



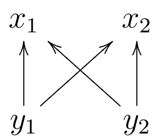
Definition 2.3. Let X_1 and X_2 be objects of a category $\mathbf{C}$. Then an object $X_1 \times X_2$ of $\mathbf{C}$ is said to be a *product* of X_1 and X_2 in the category $\mathbf{C}$ if:

(a) There are *projection* morphisms $\pi_i\colon X_1 \times X_2 \to X_i$ for $i = 1, 2$;
(b) For any object Y and for any pair of morphisms $f_i\colon Y \to X_i$ of $\mathbf{C}$, there is a unique morphism $f_1 \times f_2\colon Y \to X_1 \times X_2$ of $\mathbf{C}$ such that $(f_1 \times f_2)\pi_i = f_i$, for $i = 1, 2$.

Proposition 2.4. *Let (X, V) be a poset category. Then a product $x_1 \times x_2$ of elements x_i of X is a greatest lower bound of x_1 and x_2.*

Proof. Definition 2.3(a) says that $x_1 \times x_2$ is a lower bound for x_1 and x_2, while Definition 2.3(b) says that $x_1 \times x_2$ is greater than (or equal to) any lower bound y for x_1 and x_2. □

Example 2.5. A pair of objects in a category may not have a product: In the "2-crown" poset



there is no greatest lower bound of x_1 and x_2.

2.3. **Coproducts.** Products in the dual $\mathbf{C}^{\mathsf{op}}$ of a category $\mathbf{C}$ may be defined directly within $\mathbf{C}$.

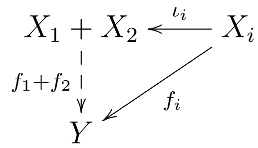
Definition 2.6. Let X_1 and X_2 be objects of a category $\mathbf{C}$. Then an object $X_1 + X_2$ of $\mathbf{C}$ is said to be a *coproduct* or *sum* of X_1 and X_2 in the category $\mathbf{C}$ if:

(a) There are *insertion* morphisms $\iota_i\colon X_i \to X_1 + X_2$ for $i = 1, 2$;

(b) For any object Y and for any pair of morphisms $f_i\colon X_i \to Y$ of **C**, there is a unique morphism $f_1 + f_2\colon X_1 + X_2 \to Y$ of **C** such that $\iota_i(f_1 + f_2) = f_i$, for $i = 1, 2$.

Here is the picture dual to (2.3):

(2.4)



Proposition 2.7. *Let* (X, V) *be a poset category. Then a coproduct or sum* x_1+x_2 *of elements* x_i *of* X *is a least upper bound of* x_1 *and* x_2*.*

Proposition 2.8. *For sets* X_1 *and* X_2*, their coproduct in the category* **Set** *is the* disjoint union

$$X_1 + X_2 = \{(x_1, 1) \mid x_1 \in X_1\} \cup \{(x_2, 2) \mid x_2 \in X_2\}$$

with insertions $\iota_i\colon X_i \to X_1 + X_2; x_i \mapsto (x_i, i)$ *for* $i = 1, 2$*.*

Proposition 2.9. *Let* U *and* V *be real vector spaces. Their coproduct in the category* **Lin** *is the* direct sum $U \oplus V = U \times V$ *with insertions* $\iota_U\colon U \to U\oplus V; u \mapsto (u, 0)$ *and* $\iota_V\colon V \to U\oplus V; v \mapsto (0, v)$*. Thus given a vector space* W *and linear transformations* $f\colon U \to W$*,* $g\colon V \to W$*, there is a unique linear transformation* $f \oplus g\colon U \oplus V \to W$ *such that* $\iota_U(f \oplus g) = f$ *and* $\iota_V(f \oplus g) = g$*.*

Remark 2.10. (a) Note the use of the special notation $f \oplus g$ rather than $f + g$ in Proposition 2.9, to avoid clashing with the established notation $f + g\colon U \to W; u \mapsto uf + ug$ for given linear transformations $f\colon U \to W$ and $g\colon U \to W$.

(b) In the context of Proposition 2.9, elements (u, v) of $U\times V$ are often written in the form $u \oplus v$.

2.4. Semilattices and lattices.

Definition 2.11. Let $(X, \leq)$ be a poset.

(a) The poset is a *meet semilattice* if for all x, y in X, the greatest lower bound or *meet* $x \wedge y$ exists.
(b) The poset is a *join semilattice* if for all x, y in X, the least upper bound or *join* $x \vee y$ exists.
(c) The poset is a *lattice* if it is both a meet and join semilattice.

Example 2.12. The divisibility poset $(\mathbb{N}, |\,)$ is a lattice. For positive integers m and n, the greatest lower bound $m\wedge n$ is the greatest common divisor, while the least upper bound $m\vee n$ is the least common multiple.

Example 2.13. The real line $(\mathbb{R}, \leq)$ is a lattice. For real numbers x and y, the greatest lower bound $x \wedge y$ is the minimum, while the least upper bound $x \vee y$ is the maximum.

2.4.1. *Algebraic characterization of semilattices and lattices.*

Definition 2.14. Let $(S, \cdot)$ be a semigroup.

(a) The semigroup is *commutative* if $\forall\ x, y \in S\ ,\ xy = yx$.
(b) The semigroup is *idempotent* if $\forall\ x \in S\ ,\ xx = x$.
(c) The semigroup is a *semilattice* if it is both idempotent and commutative.

Proposition 2.15. *(a) Let $(S, \wedge)$ be a semilattice. Defining*

$$x \leq^{\uparrow}_{\wedge} y \quad \Leftrightarrow \quad x \wedge y = x$$

yields a meet semilattice $(S, \leq^{\uparrow}_{\wedge})$ or $(S, \leq_{\wedge})$.
(b) Let $(S, \vee)$ be a semilattice. Defining

$$x \leq^{\downarrow}_{\vee} y \quad \Leftrightarrow \quad x \vee y = y$$

yields a join semilattice $(S, \leq^{\downarrow}_{\vee})$ or $(S, \leq_{\vee})$.

Remark 2.16. If $(S, \cdot)$ is a semilattice, the **meet** semilattice ordering of Proposition 2.15(a) places the product $x \cdot y$ **below** its arguments x, y, while the **join** semilattice ordering of Proposition 2.15(b) places the product $x \cdot y$ **above** its arguments x, y. In other words, the semigroup product $x \cdot y$ becomes a **product** in the **meet** semilattice poset category, and a **coproduct** in the **join** semilattice poset category.

Proposition 2.17. *Let $(S, \wedge)$ and $(S, \vee)$ be semilattices. Then the order relations $\leq^{\uparrow}_{\wedge}$ and $\leq^{\downarrow}_{\vee}$ on S coincide if and only if the* absorption laws

$$x \vee (x \wedge y) = x = x \wedge (x \vee y) \tag{2.5}$$

are satisfied.

Given Proposition 2.17, lattices may be characterized algebraically as sets $(L, \vee, \wedge)$ with two semilattice structures $(L, \vee)$ and $(L, \wedge)$ such that the absorption laws are satisfied (Exercise 18).

2.4.2. *Bounded semilattices and lattices.*

Definition 2.18. Let $(X, \leq)$ be a poset.

(a) An element b of X is an *upper bound* if $\forall\ x \in X\ ,\ x \leq b$. If an upper bound b exists, the poset is *bounded above* by b.
(b) An element b of X is a *lower bound* if $\forall\ x \in X\ ,\ b \leq x$. If a lower bound b exists, the poset is *bounded below* by b.

Proposition 2.19. *Let $(S, \cdot, e)$ be a monoid such that $(S, \cdot)$ is a semilattice. Then the identity element e is an upper bound for the meet semilattice $(S, \leq_{\cdot}^{\uparrow})$ and a lower bound for the join semilattice $(S, \leq_{\cdot}^{\downarrow})$.*

Definition 2.20. A *bounded lattice* $(L, \vee, \wedge, 0, 1)$ is a lattice $(L, \vee, \wedge)$ with monoids $(L, \vee, 0)$ and $(L, \wedge, 1)$.

2.5. **Heyting algebras.** Heyting algebras offer poset categories with an analogue of the Currying (2.1).

Definition 2.21. A *Heyting algebra* $(H, \vee, \wedge, \backslash, 0, 1)$ is defined as a bounded lattice $(H, \vee, \wedge, \backslash, 0, 1)$ with an additional binary operation $\backslash$ such that the condition

$$\forall\ x, y, z \in H\ ,\ \ x \wedge y \leq z\ \ \Leftrightarrow x \leq y\backslash z \tag{2.6}$$

is satisfied.

Example 2.22. A *Boolean algebra* $(B, \vee, \wedge, \neg, 0, 1)$ is defined as a bounded lattice $(B, \vee, \wedge, 0, 1)$ satisfying the *distributive laws*

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \quad \text{and} \quad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

together with a *complementation* $\neg x$ satisfying the *complementation laws*

$$x \wedge (\neg x) = 0 \quad \text{and} \quad x \vee (\neg x) = 1\,.$$

Let $(B, \vee, \wedge, \neg, 0, 1)$ be a Boolean algebra, for example the power set $\left(2^X, \cup, \cap, \overline{}, \emptyset, X\right)$ of a set X. Define the *implication* $y \to z = (\neg y) \vee z$. Then $(B, \vee, \wedge, \to, 0, 1)$ is a Heyting algebra.

Example 2.23. Let $I = [0, 1]$ be the closed unit interval in the real line. For elements y, z of I, define

$$y\backslash z = \begin{cases} 1 & \text{if } y \leq z\,; \\ z & \text{otherwise.} \end{cases}$$

Then with the maximum and minimum operations of Example 2.13, one obtains a Heyting algebra $(I, \vee, \wedge, \backslash, 0, 1)$.

3. Exercises

(1) Let M be a monoid. Show that the multiplication in M is determined completely by the category $\underline{\underline{M}}$ of Example 1.4.
(2) Show that a relation V on X is transitive iff $V \circ V \subseteq V$.
(3) Verify the claims of Example 1.6.
(4) Prove Proposition 1.7
(5) Prove Proposition 1.9
(6) Starting with the divisibility preorder $(\mathbb{Z}, |\,)$, determine which ordered set is produced by Proposition 1.9(b).
(7) If $\mathbf{C}$ is a category, confirm that Definition 1.10 defines $\mathbf{C}^{\mathsf{op}}$ as a category.
(8) Let G be a group, with corresponding category $\underline{\underline{G}}$ as in Example 1.4. Let G^{op} be the group determined by the dual category $\underline{\underline{G}}^{\mathsf{op}}$ as in Exercise 1. Show that the groups G and G^{op} are isomorphic.
(9) Prove Proposition 1.13.
(10) Let $f \colon U \to V$ be a linear transformation of real vector spaces. If f is a monomorphism in the category **Lin**, show that it is right invertible. [Hint: A linear transformation $g \colon \mathbb{R} \to U$ is uniquely specified by the element $1g$ in U.]
(11) Prove Proposition 1.15.
(12) Let X_1 and X_2 be sets. Give an example to show that the projection $\pi_2 \colon X_1 \times X_2 \to X_2$ need not be an epimorphism.
(13) (a) If X_1 and X_2 are groups, show that $X_1 \times X_2$, with componentwise structure, is a group.
  (b) If X_1 and X_2 are fields, show that $X_1 \times X_2$, with componentwise structure, is not a field.
(14) Prove the existence of the isomorphism (2.2), when the spaces U, V, W have respective bases X, Y, Z.
(15) A set X is said to be *pointed* if it has a special element 0_X. A function $f \colon X \to Y$ between pointed sets is called a *pointed morphism* if $0_X f = 0_Y$. Take $\mathbf{Set}^*$ to be the category of pointed sets and morphisms. For pointed sets X and Y, the morphism set $\mathbf{Set}^*(X, Y)$ is pointed, with $0_{\mathbf{Set}^*(X,Y)}$ as the function having constant value 0_Y. Now let X, Y, Z be finite pointed sets, with respective sizes $1 + l, 1 + m, 1 + n$.
  (a) What is the size of the pointed set $\mathbf{Set}^*(Y, Z)$?
  (b) If there is a Currying isomorphism

$$\mathbf{Set}^*(X \otimes Y, Z) \cong \mathbf{Set}^*(X, \mathbf{Set}^*(Y, Z)),$$

  what is the size of the pointed set $X \otimes Y$?

(16) Prove Proposition 2.8.
(17) Prove Proposition 2.9.
(18) Prove Proposition 2.17.
(19) Let X be a set. Show that the poset $(2^X, \subseteq)$ is a lattice, with intersection as the meet and with union as the join.
(20) Let U be a real vector space. Let $\mathrm{Sb}\,U$ denote the set of subspaces of U.
  (a) Show that the poset $(\mathrm{Sb}\,U, \subseteq)$ is a lattice, with intersection as the meet.
  (b) How is the join of two subspaces determined?
(21) A lattice L is said to be *distributive* if $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for all x, y, z in L.
  (a) Show that L is distributive if and only if $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for all x, y, z in L.
  (b) Show that for each set X, the lattice $(2^X, \subseteq)$ is distributive.
  (c) Show that $(\mathbb{N}, |\,)$ is distributive.
  (d) Show that $(\mathbb{R}, \leq)$ is distributive.
  (e) Show that $(\mathrm{Sb}\,U, \subseteq)$ (as in Exercise 20) is not distributive if $\dim U > 1$.
(22) Verify Proposition 2.19.
(23) Check the Currying condition (2.6) in the Boolean algebra context of Example 2.22
(24) Check the Currying condition (2.6) in the real closed interval context of Example 2.23
(25) Let $(X, \mathcal{T})$ be a topological space, with set $\mathcal{T}$ of open sets. Show that a binary operation $\backslash$ may be defined on $\mathcal{T}$ to yield a Heyting algebra $(\mathcal{T}, \cup, \cap, \backslash, \emptyset, X)$. For open subsets U and V of X, what name do topologists give to the open set $U \backslash V$?