FUNCTORS AND ADJUNCTIONS

Abstract. Graphs, quivers, natural transformations, adjunctions, Galois connections, Galois theory.

1. Functors

1.1. Graph maps.

1.1.1. *Quivers.* Quivers generalize directed graphs, and may be viewed as categories without a composition structure.

Definition 1.1. A *quiver* $C = (C_0, C_1, \partial_0, \partial_1)$ (or "graph") consists of two classes C_0, C_1 and two maps $\partial_0 \colon C_1 \to C_0$, $\partial_1 \colon C_1 \to C_0$.

(a) Elements of C_0 are called *vertices*, *points*, or *objects*.
(b) Elements of C_1 are called *edges*, *arrows*, or *morphisms*.
(c) The map ∂_0 is variously called the *tail* or *domain map.*
(d) The map ∂_1 is variously called the *head* or *codomain map.*

An edge f is often depicted in the form $f \colon x \to y$ or $x \xrightarrow{f} y$ to indicate that $f^{\partial_0} = x$ and $f^{\partial_1} = y$. For a given pair (x, y) of vertices, set

$$C(x,y) = \{f \in C_1 \mid f^{\partial_0} = x\,,\, f^{\partial_1} = y\}. \tag{1.1}$$

The quiver C is said to be *small* if the classes C_0 and C_1 are sets. The quiver is *locally small* if the class (1.1) is a set for each pair (x, y) of vertices. Usually, quivers are implicitly assumed to be locally small.

Example 1.2. Let (V, E) be a directed graph. Then the maps

$$\partial_i \colon E \to V; (v_0, v_1) \mapsto v_i$$

for $i = 0, 1$ yield a quiver $(V, E, \partial_0, \partial_1)$.

1.1.2. *Graph maps.*

Definition 1.3. A *graph map* $F : D \to C$ from a quiver D to a quiver C consists of two functions, a *vertex map* or *object part* $F_0 \colon D_0 \to C_0$ and an *edge map* or *morphism part* $F_1 \colon D_1 \to C_1$, such that for each pair x, y of vertices of D, the map F_1 restricts to

$$F_1 : D(x,y) \to C(xF_0, yF_0). \tag{1.2}$$

The respective suffices 0 and 1 on the object and morphism parts are usually suppressed.

Example 1.4. Let (V_i, E_i) be directed graphs for $i = 1, 2$, having corresponding relational structures $(V_i, \{\eta\})$ of signature $\sigma\colon \{\eta\} \to \{2\}$ given by the set η of directed edges. Then a graph homomorphism f, a homomorphism

$$f\colon (V_1, \{\eta\}) \to (V_2, \{\eta\})$$

of relational structures, yields a graph map F with

$$F_0\colon V_1 \to V_2; v \mapsto vf$$

and $F_1\colon E_1 \to E_2; (v, v') \mapsto (vf, v'f)$. Note that the domains of the restrictions (1.2) are either singletons or empty in this case.

Example 1.5. Let D be a quiver.

(a) The *identity map* $1_D : D \to D$ on D comprises the respective identity maps 1_{D_0} and 1_{D_1} on the vertex and edge classes.
(b) If C is a category, and c is an object of C, then the *constant map* $[c] : D \to C$ takes each vertex of the quiver D to c and each arrow of the quiver D to 1_c.

1.2. **Natural transformations.**

1.2.1. *Commuting diagrams.*

Definition 1.6. Let D be a quiver.

(a) A pair (f, g) of edges in D is said to be *composable* if $f^{\partial_1} = g^{\partial_0}$, i.e., if the head of f is the tail of g:

$$f^{\partial_0} \xrightarrow{f} (f^{\partial_1} = g^{\partial_0}) \xrightarrow{g} g^{\partial_1} .$$

(b) A (*non-trivial*) *path* in D is a non-empty sequence $e_1, \dots, e_l$ of edges such that each pair (e_1, e_2), $\dots$, (e_{l-1}, e_l) is composable.

Definition 1.7. Let C be a category.

(a) A *diagram* in C is a graph map $F\colon D \to C$ with codomain C.
(d) The diagram is said to *commute* if for each pair

$$(e_1, \dots, e_l)\,, \quad (f_1, \dots, f_m)$$

of paths in D with common starting point $e_1^{\partial_0} = f_1^{\partial_0}$ and end point $e_l^{\partial_1} = f_m^{\partial_1}$, the composite morphisms

$$e_1^F \dots e_l^F \text{ and } f_1^F \dots f_m^F$$

in C agree.

1.2.2. *Natural transformations.* Given two diagrams $F : D \to C$ and $G : D \to C$ with common domain quiver D and codomain category C, a *natural transformation* $\tau : F \to G$ is a vector having a component $\tau_x : xF \to xG$ in $C(xF, xG)$ for each vertex x of D, such that the *naturality property* $f^F \tau_y = \tau_x f^G$ is satisfied for each edge $f : x \to y$ of D. The naturality corresponds to the commuting of the diagram in C on the right-hand side of the picture

(1.3)

$$\boxed{\text{In } D} \qquad \begin{array}{c} x \\ {\scriptstyle f}\big\downarrow \\ y \end{array} \qquad \Bigg| \qquad \begin{array}{ccc} xF & \xrightarrow{\tau_x} & xG \\ {\scriptstyle f^F}\big\downarrow & & \big\downarrow{\scriptstyle f^G} \\ yF & \xrightarrow{\tau_y} & yG \end{array} \qquad \boxed{\text{In } C}$$

for every arrow $f : x \to y$ in D displayed on the left-hand side of the picture.

Example 1.8. Let D be a quiver. If $h : a \to b$ is a morphism of a category C, then the *constant* natural transformation $[h] : [a] \to [b]$ has the morphism $h : a \to b$ as its component at each vertex x of D.

1.3. **Functors.**

Definition 1.9. Let D and C be categories.

(a) A (*covariant*) *functor* $F : D \to C$ is a graph map satisfying the *functoriality properties* $1_x F = 1_{xF}$ for all objects x of D and

$$(fg)^F = f^F g^F \tag{1.4}$$

for all composable pairs (f, g) of D.

(b) A *contravariant functor* $F : D \to C$ is a covariant functor from D to C^{op}.

Example 1.10. Let (X, V) and (X', V') be poset categories.

(a) The object part of a functor $F\colon (X, V) \to (X', V')$ is an *order-preserving* map, i.e. $(x, y) \in V$ implies $(xF, yF) \in V'$.

(b) The object part of a contravariant functor $F\colon (X, V) \to (X', V')$ is an *order-reversing* map, i.e. $(x, y) \in V$ implies $(yF, xF) \in V'$.

Example 1.11 (Forgetful functors)**.** Let $\mathbf{C}$ be a category of algebras and homomorphisms. Then the *forgetful functor* $G\colon \mathbf{C} \to \mathbf{Set}$ assigns the underlying set to each algebra, and the underlying function to each homomorphism.

Let C be a category. Then the *identity functor* 1_C on C has object part 1_{C_0} and morphism part 1_{C_1}. If $F\colon D \to C$ and $F'\colon C \to B$ are functors, then so is the composite $FF'\colon D \to B$.

2. Adjunctions

If $F\colon D \to C$ and $G\colon C \to D$ are mutually inverse functions, then $1_D = FG$ and $GF = 1_C$. The concept of an adjunction provides an analogous relationship for functors.

2.1. **Left and right adjoints.**

Let C and D be categories. Then an *adjunction* $(F, G, \eta, \varepsilon)$ consists of the following data:

- a *left adjoint* functor $F : D \to C$,
- a *right adjoint* functor $G : C \to D$,
- a *unit* natural transformation $\eta : 1_D \to FG$, and
- a *counit* natural transformation $\varepsilon : GF \to 1_C$,

such that

$$\eta_x^F \varepsilon_{xF} = 1_{xF} \quad \text{and} \quad \eta_{yG}\varepsilon_y^G = 1_{yG} \tag{2.1}$$

for all objects x of D and for all objects y of C.

Such an adjunction is often summarized by the isomorphism

$$F \text{ on left} \longrightarrow \boxed{C(xF, y) \cong D(x, yG)} \longleftarrow G \text{ on right} \tag{2.2}$$

for objects x of D and y of C, under which a morphism $g\colon xF \to y$ maps to $\eta_x g^G$, while a morphism $f\colon x \to yG$ maps to $f^F\varepsilon_y$. In particular, η_x corresponds to 1_{xF} and ε_y corresponds to 1_{yG}.

2.1.1. *Free algebras.*

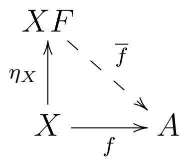
Example 2.1. For a characteristic example of an adjunction, take C to be the category **Mon** of monoids and monoid homomorphisms, with D as the category **Set** of sets. The right adjoint is the underlying set functor $G : \mathbf{Mon} \to \mathbf{Set}$, while the left adjoint $F\colon \mathbf{Set} \to \mathbf{Mon}$ takes a set X, considered as an alphabet, to the free monoid X^* of words in the alphabet X (with the empty word as the identity element). Words are multiplied by concatenation. The general adjunction (2.2) takes the form

$$\mathbf{Mon}(XF, M) \cong \mathbf{Set}(X, MG) \tag{2.3}$$

for a monoid M. The component $\eta_X\colon X \to X^*$ at a set X embeds letters (elements) from X into X^* as one-letter words. For a monoid M, the counit $\varepsilon_M : M^* \to M$ takes a word in the alphabet M to the product of its letters computed in the monoid M. Under the isomorphism (2.3), a monoid homomorphism $g\colon XF \to M$ or $g\colon X^* \to M$ is mapped to its restriction to the set X of one-letter words. Conversely, a function $f\colon X \to M$ from a set X to (the underlying set of) a monoid M is mapped to its canonical extension to a monoid homomorphism from X^* to M.

Example 2.2. The constructions of Example 2.1 carry over when the category **Mon** of monoids is replaced by a more general category C of algebras and homomorphisms. Suppose that for each set X, a free algebra XF has been constructed. Then for each algebra A from C, and for each function $f\colon X \to A$, there is a unique homomorphism $\overline{f}\colon XF \to A$ such that $\eta_X \overline{f} = f$. Up to now, this situation has been illustrated by diagrams of the following form:

(2.4)



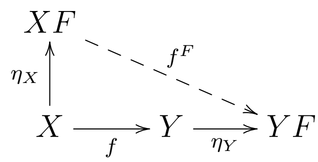
Now while the arrow $\overline{f}$ is a morphism in **C**, the arrows η_X and f are morphisms in **Set**, although their respective codomains are indicated in (2.4) by objects from the category **C**. For added precision, diagrams like (2.4) are replaced by the adjunction isomorphism

$$\mathbf{C}(XF, A) \cong \mathbf{Set}(X, AG) \tag{2.5}$$

featuring the forgetful functor $G\colon \mathbf{C} \to \mathbf{Set}$ of Example 1.11. The function η_X is written as the morphism $\eta_X\colon X \to XFG$ in **Set**, the component at X of the natural transformation η. The extension from f to $\overline{f}$ in diagram (2.4) becomes an extension from $f\colon X \to AG$ to $\overline{f}\colon XF \to A$, the application

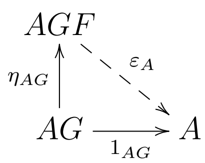
$$\overline{f} \longmapsfrom f$$

or $f \mapsto f^F \varepsilon_A$ of the isomorphism (2.5). The image $f^F\colon XF \to YF$ of a function $f\colon X \to Y$ under the *free algebra functor* F is given from (2.4) as the unique homomorphic extension $\overline{f\eta_Y}$ of the function $f\eta_Y\colon X \to YFG$:



The counit component $\varepsilon_A\colon AGF \to A$ at an algebra A is given by:

(2.6)



— compare the left hand side of (2.1), and see Exercise 9.

2.2. **Currying and tensor products.**

2.2.1. *Currying revisited.*

Proposition 2.3. *Let Y be a set. Consider the functors*

$$F_Y \colon \mathbf{Set} \to \mathbf{Set}; Z \mapsto Z \times Y$$

(object part) and

$$G_Y \colon \mathbf{Set} \to \mathbf{Set}; (f \colon X \to X') \mapsto (X^Y \to X'^Y; g \mapsto gf)$$

(morphism part). Then the currying isomorphism

$$\mathbf{Set}(Z \times Y, X) \cong \mathbf{Set}(Z, \mathbf{Set}(Y, X)) \tag{2.7}$$

becomes the adjoint relationship

$$\mathbf{Set}(ZF_Y, X) \cong \mathbf{Set}(Z, XG_Y)$$

between F_X and G_Y.

2.2.2. *Bilinear functions and tensor products.* Let S be a commutative, unital ring, with the corresponding category $\underline{\underline{S}}$ of S-modules.

Definition 2.4. Consider S-modules X, Y, Z.

(a) A function $f \in \mathbf{Set}(Z \times Y, X)$ is *bilinear* if its curried version in $\mathbf{Set}(Z, \mathbf{Set}(Y, X))$ actually lies in $\underline{\underline{S}}(Z, \underline{\underline{S}}(Y, X))$. In other words, the curried functions

$$f_y \colon Z \to X; z \mapsto (y, z)f$$

for $y \in Y$ and

$$f_z \colon Y \to X; y \mapsto (y, z)f$$

for $z \in Z$ are both linear (i.e., S-module homomorphisms).

(b) The set of bilinear functions from $Z \times Y$ to X is written as $\underline{\underline{S}}(Z, Y; X)$.

For the general proof of the following proposition, see Section III.3.6 in Post-Modern Algebra. For the case where X, Y, Z are vector spaces over a field S, compare Exercise 14 in the Categories notes.

Proposition 2.5. *Let Y be an S-module. Then the functor*

$$G_Y \colon \underline{\underline{S}} \to \underline{\underline{S}}; (f \colon X \to X') \mapsto (\underline{\underline{S}}(Y, X) \to \underline{\underline{S}}(Y, X'); g \mapsto gf)$$

(morphism part) has a left adjoint $F_Y \colon \underline{\underline{S}} \to \underline{\underline{S}}; Z \mapsto Z \otimes Y$.

Definition 2.6. The module $Z \otimes Y$ is called the (S-*module*) *tensor product* of Z and Y.

Corollary 2.7. *Let X, Y, Z be S-modules. Then each bilinear function $f \colon Z \times Y \to X$ determines a unique linear function $f \colon Z \otimes Y \to X$.*

2.3. **Galois connections.** A *Galois connection* is an adjunction

$$(F\colon D \to C, G\colon C \to D, \eta, \varepsilon) \tag{2.8}$$

where the categories C and D are poset categories $(C. \leq)$, $(D, \leq)$. The isomorphism relationship (2.2) reduces to

$$\forall\, x \in D\,,\ \forall\, y \in C\,, \quad x^F \leq y \ \Leftrightarrow\ x \leq y^G\,.$$

The existence of the unit and counit reduces to

$$\begin{cases} \text{(a)} & \forall\, x \in D\,,\ x \leq x^{FG}; \\ \text{(b)} & \forall\, y \in C\,,\ y^{GF} \leq y. \end{cases}$$

The identities (2.1) reduce to

$$\begin{cases} \text{(a)} & \forall\, x \in D\,,\ x^F = x^{FGF}; \\ \text{(b)} & \forall\, y \in C\,,\ y^{GFG} = y^G. \end{cases} \tag{2.9}$$

Example 2.8. Let $f\colon A \to B$ be a function. Define

$$F\colon (2^A, \subseteq) \to (2^B, \subseteq); X \mapsto Xf$$

and

$$G\colon (2^B, \subseteq) \to (2^A, \subseteq); Y \mapsto f^{-1}Y$$

with the *inverse image* $f^{-1}Y = \{x \in A \mid xf \in Y\}$. Since

$$Xf \subseteq Y \ \Leftrightarrow\ X \subseteq f^{-1}Y\,,$$

the functors F and G yield a Galois connection.

2.3.1. *Galois correspondence.* In a Galois connection (2.8), elements of the subsets DF of C and CG of D are described as *closed*. One obtains induced posets $(DF, \leq)$ and $(CG, \leq)$. By (2.9), the restricted functors $F\colon CG \to DF$ and $G\colon DF \to CG$ yield mutually inverse isomorphisms. The isomorphism

$$(DF, \leq) \cong (CG, \leq)$$

is known as a *Galois correspondence.*

2.3.2. *Polarities.*

Definition 2.9. Let I and J be sets. A subset α of $I \times J$ is known as a *relation between* the sets I and J. One often writes $x\ \alpha\ y$ for $(x, y) \in \alpha$.

Example 2.10. Let n be a positive integer. The *orthogonality relation*

$$\mathbf{x} \perp \mathbf{u} \ \Leftrightarrow\ \mathbf{xu} = 0$$

is a relation between the space $\mathbb{R}^n_1$ of n-dimensional row vectors $\mathbf{x}$ and the space $\mathbb{R}^1_n$ of n-dimensional column vectors $\mathbf{u}$.

Suppose that α is a relation between I and J. Consider the poset categories $(2^I, \subseteq)$ and $(2^J, \supseteq)$. Define

$$F\colon (2^I, \subseteq) \to (2^J, \supseteq); X \mapsto \{y \in J \mid \forall\, x \in X,\ x \alpha y\}$$

and

$$G\colon (2^J, \supseteq) \to (2^I, \subseteq); Y \mapsto \{x \in I \mid \forall\, y \in Y,\ x \alpha y\}$$

Since

$$X^F \supseteq Y \ \Leftrightarrow\ [\,\forall\, x \in X\,,\ \forall\, y \in Y\,,\ x\ \alpha\ y\,]\ \Leftrightarrow\ X \subseteq Y^G\,,$$

the functors F and G form a Galois connection, known as the *polarity* determined by the relation α.

In the context of the following definition, it is convenient to use a different notation for the adjoint functors.

Definition 2.11. Let G be a group, and let X be a G-set. Consider the subset

$$\{(x, g) \in X \times G \mid xg = x\} \tag{2.10}$$

of $X \times G$ used in the proof of Burnside's Lemma as a relation between X and G. Write $X_g = \{x \in X \mid xg = x\}$ for $g \in G$.

(a) The right adjoint

$$F\colon (2^G, \supseteq) \to (2^X, \subseteq); H \mapsto \bigcap_{h \in H} X_h$$

in the polarity determined by (2.10) is known as the *fixed-point functor.*

(a) The left adjoint

$$S\colon (2^X, \subseteq) \to (2^G, \supseteq); Y \mapsto \bigcap_{y \in Y} G_y$$

in the polarity determined by (2.10) is known as the *stabilizer functor.*

There is an interesting correspondence between G-subsets of X and normal subgroups of G in the context of Definition 2.11.

Proposition 2.12. *Let G be a group, and let X be a G-set.*

(a) *If Y is a G-subset of X, then Y^S is a normal subgroup of G.*

(b) *If N is a normal subgroup of G, then N^F is a G-subset of X.*

Proof. (a) For $g \in G$, $n \in Y^S$, and $y \in Y$, one has

$$yg^{-1}ng = yg^{-1} = y\,,$$

so $g^{-1}ng \in Y^S$.

(b) For $g \in G$, $y \in N^F$, and $n \in N$, one has

$$ygn = ygng^{-1}g = yg\,,$$

so $yg \in N^F$. □

2.4. **Galois theory.** The classical application of the polarity that was introduced in Definition 2.11 is to the action of a so-called "Galois group" on a field.

Definition 2.13. Let A be a field that contains a field K as a unital subring.

(a) Let $G = \mathrm{Aut}_K A$ denote the group of all ring automorphisms of A that leave each element of K fixed. Then $G = \mathrm{Aut}_K A$ is known as the *Galois group* of A over K.
(b) A field C that contains K, and is a subring of A, is known as an *intermediate field*.

Consider the polarity

$$F : (2^G, \supseteq) \leftrightarrows (2^A, \subseteq) : S$$

introduced in Definition 2.11 for the G-set A. Note that $K \subseteq G^F$, by the definition of the Galois group G. For each subgroup H of G, the fixed point set H^F is an intermediate field. As usual, for each subset X of A, the stabilizer X^S is a subgroup of G.

Lemma 2.14. *The base field K is a closed subset of A if and only if $K = G^F$.*

Proof. If $K = G^F$, its closure is immediate. Conversely, suppose that $K = Y^F$ for $G \supseteq Y$. Then $K \subseteq G^F \subseteq Y^F = K$. □

Identification of further closed elements relies on a dual pair of inequalities.

Proposition 2.15. *Suppose that $B \leq C$ for intermediate fields B and C with $\dim_B C$ finite. Then $|C^S \backslash B^S| \leq \dim_B C$.*

Proof. Use induction on $\dim_B C$.

- If $B = C$, the result is trivial.
- If there is an intermediate field D with $B < D < C$, then

$$|C^S \backslash B^S| = |C^S \backslash D^S| \cdot |D^S \backslash B^S| \leq (\dim_D C)(\dim_B D) = \dim_B C$$

by induction.

- One may thus assume that $C = B(t)$ for an element t of C. Let $p(X) \in B[X]$ be the minimal polynomial of t. Now
$$C^S \backslash B^S \to \{x \in C \mid p(x) = 0\}; C^S U \mapsto t^U$$
is a well-defined injection. Then
$$|C^S \backslash B^S| \leq \deg p(X) = \dim_B C$$
as required.

□

Proposition 2.16. *Suppose that* $H \geq J$ *for subgroups* H *and* J *of* G *with* $|J\backslash H|$ *finite. Then* $\dim_{HF} J^F \leq |J\backslash H|$.

Proof. Let $\{1 = U_1, U_2, \ldots U_n\}$ be a set of representatives for the right cosets of J in H. Suppose that there were an independent subset $\{a_1, \ldots, a_{n+1}\}$ of the HF-space JF. Take the $(n+1) \times n$ matrix $[a_i U_j]$ over A as the coefficient matrix of a homogeneous system of n equations in $n+1$ unknowns. Choose a non-trivial solution with a minimal number $r+1$ of non-zero components. Without loss of generality (reordering the elements $a_1, \ldots, a_{n+1}$ as required), the solution may be taken as
$$(x_1, \ldots, x_r, 1, 0, \ldots, 0) \tag{2.11}$$
in A^{n+1}.

If the statement
$$\forall\ 1 \leq i \leq r\,,\ x_i \in HF$$
were true, the first equation of the homogeneous system would yield the contradiction
$$x_1 a_1 + \ldots + x_r a_r + a_{r+1} = 0$$
to the supposed linear independence of $\{a_1, \ldots, a_{n+1}\}$. Thus
$$\exists\ 1 \leq i \leq r\,.\ x_i \notin HF$$
or
$$\exists\ V \in H\,.\ \exists\ 1 \leq i \leq r\,.\ x_i - x_i^V \neq 0\,.$$
Again without loss of generality, one may assume that
$$x_1 - x_1^V \neq 0 \tag{2.12}$$
for some element V of H.

Applying the field automorphism V to the original homogeneous system, one obtains a new homogeneous system with solution
$$(x_1^V, \ldots, x_r^V, 1, 0, \ldots, 0) \tag{2.13}$$

in A^{n+1}. Consider $\pi \in \{1, \ldots, n\}!$ with $U_j V^{-1} \in JU_{j\pi}$ or $U_{j\pi} V \in JU_j$ for $1 \leq j \leq n$. Then the j-th equation of the old system becomes the $j\pi$-th equation of the new system. In other words, the coefficient matrix of the new system is obtained from the old coefficient matrix, postmultiplying by the permutation matrix of the permutation π. Thus both (2.11) and (2.13) are solutions to the original system. Since the system is homogeneous, their difference

$$(x_1 - x_1^V, \ldots, x_r - x_r^V, 0, 0, \ldots, 0) \tag{2.14}$$

is also a solution. By (2.12), the solution (2.14) is non-trivial, but with fewer than $r+1$ non-zero components. This contradicts the minimality of $r+1$. □

The following result is the *Fundamental Theorem of Galois Theory.*

Theorem 2.17. *Let K be a field. Let A be a field containing K, such that the dimension $\dim_K A$ of A as a vector space over K is finite. Let G be the Galois group of A over K. Suppose that K is a closed subset of A.*

(a) *The group G is finite, with $|G| = \dim_K A$.*
(b) *Each subgroup of G is a closed subset of G.*
(c) *Each intermediate field C is a closed subset of A.*
(d) *The Galois correspondence yields a bijection between the set of subgroups of G and the set of intermediate fields.*
(e) *If C is an intermediate field, then $\dim_K C = |G|/|C^S|$.*
(f) *If H is a subgroup of G, then $|H| = \dim_{HF} A$.*

Proof. For each intermediate field C, one has

$$\dim_K C^{SF} = \dim_{KSF}^{SF} \leq |C^S \backslash K^S| \leq \dim_K C \leq \dim_K C^{SF} \,,$$

and $C \leq C^{SF}$, so $C = C^{SF}$ is closed and

$$\dim_K C = |C^S \backslash K^S| = |C^S \backslash G| \,.$$

In particular, $|G| = |A^S \backslash K^S| = \dim_K A$. Then for each subgroup H of G, one has

$$|H^{FS}| = |1^{FS} \backslash H^{FS}| \leq \dim_{HF} 1^F \leq |1 \backslash H| = |H| \leq |H^{FS}|$$

and also $H^{FS} \geq H$, so $H = H^{FS}$ is closed and $|H| = \dim_{HF} A$. □

Example 2.18. Let n be a positive integer. Let $\mathbb{Q}[\zeta_n]$ be the smallest subfield of $\mathbb{C}$ that contains the n-th root of unity $\zeta_n = \exp(2\pi i/n)$. Then the Galois group of $\mathbb{Q}[\zeta_n]$ over $\mathbb{Q}$ is the group $(\mathbb{Z}/_n, +, 0)^*$ of units. If n is an odd prime, the intermediate fields correspond to the divisors r of $n-1$.

3. Exercises

(1) Let C be a poset category. Show that each diagram $F\colon D \to C$ commutes.

(2) Let D be a quiver. Define $D\Pi_1$ to be the disjoint union of the class D_0 and the class $D\Pi_1^+$ of paths in D. Define $d_0\colon D\Pi_1 \to D$ to be the disjoint union of the identity $D_0 \to D_0$ and the map
$$D\Pi_1^+ \to D_0; (e_1, \dots, e_l) \mapsto e_1^{\partial_0}.$$
Define $d_1\colon D\Pi_1 \to D$ to be the disjoint union of the identity $D_0 \to D_0$ and the map
$$D\Pi_1^+ \to D_0; (e_1, \dots, e_l) \mapsto e_l^{\partial_1}.$$
Define a composition of non-trivial paths by
$$\big((e_1, \dots, e_l), (f_1, \dots, f_m)\big) \mapsto (e_1, \dots, f_m)$$
when $e_m^{\partial_1} = f_1^{\partial_0}$ in D. Show that there is a unique way to extend these partial data to yield a category $D\Pi = (D, D\Pi_1)$. This category is known as the *path category* of the quiver D.

(3) A quiver D is said to be *discrete* if D_1 is empty. Determine the path category $D\Pi$ of a discrete quiver D.

(4) Let (X, V) be a poset. An element (x, y) of V is a *covering pair* if $|\{z \in X \mid x \le z \le y\}| = 2$. Let $C(X, V)$ be the set of covering pairs of (X, V), equipped with the projections
$$\pi_i\colon C(X, V) \to X; (x_0, x_1) \mapsto x_i\,.$$
The *Hasse diagram* of (X, V) is the quiver $\big(X, C(X, V), \pi_0, \pi_1\big)$. Show that (X, V) is the path category of its Hasse diagram.

(5) For the natural transformation $[h]$ of Example 1.8, draw the naturality diagram (1.3).

(6) Let M and N be monoids, with corresponding single-object categories $\underline{\underline{M}}$ and $\underline{\underline{N}}$.
   (a) Show that a monoid homomorphism $f\colon M \to N$ yields a functor $F\colon \underline{\underline{M}} \to \underline{\underline{N}}$ with $mF = mf$ for each morphism m of $\underline{\underline{M}}$.
   (b) Show that a functor $F\colon \underline{\underline{M}} \to \underline{\underline{N}}$ yields a monoid homomorphism $f\colon M \to N$ with $mf = mF$ for each element m of M.

(7) Give an example of lattices L, L' and an order-preserving map $F\colon (L, \le) \to (L', \le)$ such that $F\colon (L, \vee, \wedge) \to (L', \vee, \wedge)$ is not a lattice homomorphism.

(8) Consider the context of Example 2.1.
   (a) For each monoid M, verify that $\eta_M \varepsilon_M^G = 1_M$.

(b) For each set X, verify that $\eta^F_X \varepsilon_{X^*} = 1_{X^*}$.

(9) In Example 2.2, draw the precise, categorical diagram that implements the left hand side of (2.1) for $y = A$, and note how it differs from the more informal diagram (2.6).

(10) In the context of Example 2.2, take $\mathbf{C}$ to be the category **Lin** of real vector spaces and linear transformations. What is the component ε_V of the counit ε at a vector space V?

(11) Consider the categories **Mon** of monoids and **Gp** of groups. Let $U\colon \mathbf{Gp} \to \mathbf{Mon}$ be the functor that forgets the inversion in a group. Recalling the notation M^* for the group of units in a monoid M, verify the adjunction relationship

$$\mathbf{Mon}(GU, M) \cong \mathbf{Gp}(G, M^*)$$

for a group G and monoid M.

(12) An adjunction $(F\colon C \to D, G\colon D \to C, \eta, \varepsilon)$ is said to provide an *equivalence* between the categories C and D if the components η_x and ε_y of the unit and counit are always isomorphisms. Let $\underline{\mathbb{N}}$ be the category with object set $\{\underline{n} \mid n \in \mathbb{N}\}$, taking all functions between the various sets $\underline{n}$ as morphisms. Let **FinSet** denote the category of finite sets and functions between them. Show that there is an equivalence between $\underline{\mathbb{N}}$ and **FinSet**.

(13) Let M be a submonoid of a monoid N, with corresponding categories $\underline{\underline{M}}$ and $\underline{\underline{N}}$ of actions. Show that there is an adjunction between induction $\uparrow^N_M\colon \underline{\underline{M}} \to \underline{\underline{N}}$ and restriction $\downarrow^N_M\colon \underline{\underline{N}} \to \underline{\underline{M}}$.

(14) Consider the context of Proposition 2.3.
(a) Identify the morphism part of F_Y.
(b) Determine the unit and counit for the adjunction between F_Y and G_Y.

(15) Prove Corollary 2.7. [Hint: Apply the composition

$$\underline{\underline{S}}(Z, Y; X) \cong \underline{\underline{S}}(Z, \underline{\underline{S}}(Y, X)) \cong \underline{\underline{S}}(Z \otimes Y, X)$$

of the Currying isomorphism with the adjointness relation from Proposition 2.5.]

(16) Let $(H, \vee, \wedge, \backslash, 0, 1)$ be a Heyting algebra, also considered as a poset category $(H, \leq)$. Then for each element y of H, show that the functors

$$S(y)\colon H \to H; x \mapsto x \wedge y$$

and

$$R(y)\colon H \to H; z \mapsto y \backslash z$$

form a Galois connection.

(17) Consider the polarity determined by the orthogonality relation of Example 2.10. Show that the closed subsets are subspaces.
(18) Let G be a group. Consider the polarity determined by the *commutation* relation $\{(x, y) \in G^2 \mid xy = yx\}$ on G.
   (a) Show that the closed elements are subgroups of G.
   (b) For the case $G = S_4$, determine which subgroups are closed.
(19) Show that $\mathbb{Q}$ is not a closed subfield of $\mathbb{R}$.
(20) Determine the Galois group of $\mathbb{Z}/_5[\sqrt{2}]$ over $\mathbb{Z}/_5$.
(21) Determine the fields intermediate between $\mathbb{Q}$ and $\mathbb{Q}[\zeta_5]$.