MATH 504 FALL 2014 PRACTICE FINAL

Each question is worth 6 points. The best 5 solutions will be taken.

(1) Determine the structure of the group of units of the monoid of residues modulo 16 under multiplication.

(2) Let R be a unital subring of $\mathbb{R}$. Consider the subset

$$R[\omega] = \left\{ \begin{bmatrix} x + \frac{y}{2} & \frac{y\sqrt{3}}{2} \\ -\frac{y\sqrt{3}}{2} & x + \frac{y}{2} \end{bmatrix} \,\middle|\, x, y \in R \right\}$$

of the matrix ring R_2^2. Show that $R[\omega]$ is an integral domain.

(3) Prove that a finite integral domain is a field.

(4) Let R be a commutative, unital ring.

  (a) Explain what it means for an ideal J of R to be *maximal*.

  (b) Explain what it means for an ideal J of R to be *prime*.

  (c) Prove that every maximal ideal is prime.

  (d) Give an example of a commutative, unital ring R and a prime ideal in R which is not maximal.

(5) Let M and N be normal subgroups of a group G. If $G = MN$, show that $G/(M \cap N) \cong G/M \times G/N$.

(6) Let G be a transitive permutation group on a set X of finite size n. Suppose that the stabilizer of an element x of X is transitive on $X \smallsetminus \{x\}$. Show that $|\, G \,|$ is divisible by $n(n-1)$.

(7) Let A and B be finite subgroups of a group G. Prove or disprove: $|AB| = |A||B|/|A \cap B|$.

(8) For a prime number p, let P be a Sylow p-subgroup of a finite group G. Let H be the subgroup of G consisting of all elements h of G such that the conjugate $\tau_h(P)$ of P coincides with P. Show that the index of H in G is congruent to 1 modulo p.