MATH 504 FALL 2014 PRACTICE MIDTERM

Write clearly, on separate paper. All questions carry equal weight.
You will receive credit for your three best answers.

(1) Let N be a normal subgroup of a group G. Show that there is a bijection

$$\{M \mid N \lhd M \lhd G\} \to \{L \mid L \lhd G/N\}; M \mapsto MN/N\,.$$

(2) For $1 < n \in \mathbb{N}$ and $0 \leq r < n$, define $f_r\colon \mathbb{C} \to \mathbb{C}; z \mapsto ze^{2\pi ir/n}$ and $c\colon \mathbb{C} \to \mathbb{C}; z \mapsto \overline{z}$.

(a) Show that $D_n = \{f_0, \ldots, f_{n-1}, f_0 \circ c, \ldots, f_{n-1} \circ c\}$ is a subgroup of $\mathbb{C}!$.

(b) Show that the group D_4 is not isomorphic to Q_8.

(3) Let I and J be ideals of a ring R. If $I + J = R$, show that $R/(I \cap J) \cong R/I \times R/J$.

(4) Consider elements A and D of the ring

$$\mathbb{Z}[i] = \left\{ \begin{bmatrix} x & -y \\ y & x \end{bmatrix} \;\middle|\; x, y \in \mathbb{Z} \right\}$$

with $D \neq 0$.

(a) Show that there are elements Q and R of $\mathbb{Z}[i]$ such that $A = DQ + R$ and $\det R < \det D$.

(b) Present an example to show that, for given A and D, the elements Q and R need not be unique.