MATH 504 FALL 2014 GRADED HOMEWORK #3

Write clearly, on separate paper. All questions carry equal weight.
You will receive credit for your three best answers.

(1) Let $(G, \cdot, e)$ be a group. Consider the group $\mathbb{Z}$ of integers. For two homomorphisms $\theta : \mathbb{Z} \to G$ and $\varphi : \mathbb{Z} \to G$, define the *componentwise product*

$$\theta \cdot \varphi : \mathbb{Z} \to G; n \mapsto \theta(n)\varphi(n) .$$

Show that G is abelian if and only if the following condition is satisfied:

For all homomorphisms

$$\theta : \mathbb{Z} \to G \text{ and } \varphi : \mathbb{Z} \to G ,$$

the componentwise product $\theta \cdot \varphi : \mathbb{Z} \to G$ is also a homomorphism.

(2) Consider the monoid $\big(\mathrm{End}(\mathbb{Z}/_2 \oplus \mathbb{Z}/_2), \circ, 1\big)$ of endomorphisms of the abelian group $\mathbb{Z}/_2 \oplus \mathbb{Z}/_2$ under composition. Show that the group of units of $\mathrm{End}(\mathbb{Z}/_2 \oplus \mathbb{Z}/_2)$ is isomorphic to the symmetric group S_3.

(3) Let R be a unital ring. Suppose that a finitely generated free R-module is the internal direct sum of submodules P and Q. Let $g\colon A \to B$ be a surjective R-module homomorphism. Let $h\colon P \to B$ be an R-module homomorphism. Show that there is an R-module homomorphism $\overline{h}\colon P \to A$ such that $g \circ \overline{h} = h$.

(4) Do the monoids $(\mathbb{Z}/_{51}, \cdot, 1)$ and $(\mathbb{Z}/_{15}, \cdot, 1) \times (\mathbb{Z}/_5, \cdot, 1)$ have isomorphic groups of units? Justify your answer.