MATH 504 FALL 2014 GRADED HOMEWORK #2

Write clearly, on separate paper. All questions carry equal weight.
You will receive credit for your three best answers.

(1) In the matrix ring $\mathbb{Z}_4^4$, set

$$i = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \ j = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \ k = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

and

$$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$$

(as usual, identifying scalars z with diagonal matrices zI_4).

(a) Show that Q_8 forms a nonabelian subgroup of the group of units of the monoid $(\mathbb{Z}_4^4, \cdot, I_4)$.

(b) Show that each subgroup of Q_8 is normal.

(2) (a) In a group G, consider elements x and y of coprime finite orders a and b. Suppose that $xy = yx$. Show that there is a group isomorphism

$$\mathbb{Z}/_a \times \mathbb{Z}/_b \to \langle xy \rangle ; (r, s) \mapsto x^r y^s .$$

(b) In the symmetric group S_n, consider disjoint cycles α and β of respective lengths a and b, with a and b coprime. Show that $\alpha \circ \beta$ has order ab.

(3) Suppose that $xx = x$ for each element x of a ring R.

(a) Show that $x + x = 0$ for all x in R.

(b) Show that R is commutative.

(4) Let x be an element of a unital ring R. Write

$$RxR = \left\{ \sum_{j=0}^{n} s_j x t_j \;\middle|\; n \in \mathbb{N};\ s_j, t_j \in R \right\} .$$

Show that $xR + Rx + RxR$ is an ideal of R.