MATH 302 SPRING 2012 PRACTICE FINAL

Write clearly. Box or underline your final answers to computational questions. All questions carry equal weight.

1. Let x be an element of a unital ring R. If $x^7 = 0$, show that $1 + x$ is an invertible element of R.
2. Exhibit a ring isomorphism $\left(\mathbb{Z}/_{21}\right)/(7\mathbb{Z}/_{21}) \cong \mathbb{Z}/_7$.
3. The graphs of two cubic functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ each pass through the points $(0, 10), (1, 12), (2, 8)$ and $(3, 1)$. Prove that $f = g$.
4. Find a real number a such that $\mathbb{Q}(\sqrt{5}, \sqrt{7}) = \mathbb{Q}(a)$. Justify your answer.
5. Find a monic polynomial $p(X)$ in $\mathbb{Q}[X]$ such that the quotient field $\mathbb{Q}[X]/p(X)\mathbb{Q}[X]$ is isomorphic to the field $\mathbb{Q}\left(\sqrt{1+\sqrt{3}}\right)$.
6. Over the field $\mathbb{Z}/_2$, there are 2 irreducible polynomials of degree 1, 1 of degree 2, 2 of degree 3, and 3 of degree 4. How many irreducible polynomials of degree 5 are there over $\mathbb{Z}/_2$?