MATH 301 SPRING 2017 PRACTICE FINAL

Write clearly, on separate paper. All questions carry equal weight.
You will receive credit for your six best answers.

(1) Prove or disprove the following statement:

The set

$$S = \{f \colon \mathbb{N} \to \mathbb{N} \mid \text{The image } f(\mathbb{N}) \text{ is infinite}\}$$

is a semigroup of functions on the base set $\mathbb{N}$.

(2) For a positive integer d, define $\varphi(d) = |(\mathbb{Z}/_d, \cdot, 1)^*|$.

(a) Show that $\varphi(p) = p - 1$ if p is prime.

(b) Show that $\varphi(mn) = \varphi(m)\varphi(n)$ if m and n are coprime.

(3) Show that

$$\{\mathrm{id}_D, (N\ S)(E\ W), (N\ E)(W\ S), (N\ W)(S\ E)\}$$

is a normal subgroup of the group of all permutations of the set $D = \{N, S, E, W\}$.

(4) Let X be a subset of a group G. Let L be the intersection of all the normal subgroups of G that contain X. Show that L is a normal subgroup of G.

(5) Let x and y be elements of a group G. Let x have finite order a, and y have finite order b, with $\gcd(a, b) = 1$. If $xy = yx$, show that xy has order ab.

(6) Let R be a commutative ring. Show that

$$N = \{x \ \text{ in } R \mid \exists\, n \in \mathbb{N} \,.\, x^n = 0\}$$

is an ideal of R.

(7) Let x be an element of a unital ring R. If $x^7 = 0$, show that $1 + x$ is an invertible element of $(R, \cdot, 1)$.

(8) Prove the identity

$$\sum_{r=0}^{n} (-1)^r \binom{n}{r} = 0$$

for positive integers n.