MATH 301B SPRING 2016 PRACTICE FINAL

Write clearly, on separate paper. All questions carry equal weight.
You will receive credit for your six best answers.

(1) Prove or disprove the following statement:
A function $f : X \to Y$ is surjective if and only if there is a function $s : Y \to X$ such that $f \circ s = \mathrm{id}_Y$.

(2) For a positive integer d, define $\varphi(d) = |(\mathbb{Z}/_d, \cdot, 1)^*|$.
(a) Show that $\varphi(p) = p - 1$ if p is prime.
(b) Show that $\varphi(mn) = \varphi(m)\varphi(n)$ if m and n are coprime.

(3) Let e be an element of a group $(G, \cdot)$.
(a) Show that the set G forms a group under the multiplication
$$* : G \times G \to G; (x, y) \mapsto xe^{-1}y.$$
(b) Show that the group $(G, *)$ with multiplication given in (a) is isomorphic with the original group structure $(G, \cdot)$ on G.

(4) Let X be a subset of a group G. Let L be the intersection of all the subgroups of G that contain X. Show that L is a subgroup of G.

(5) Let x and y be elements of a group G. Let x have finite order a, and y have finite order b, with $\gcd(a, b) = 1$. If $xy = yx$, show that xy has order ab.

(6) Let X be a subset of a ring R. Show that
$$C = \{r \text{ in } R \mid rx = xr \text{ for all } x \text{ in } X\}$$
is a subring of R.

(7) Let x be an element of a unital ring R. If $x^7 = 0$, show that $1 - x$ is an invertible element of $(R, \cdot, 1)$.

(8) Prove the identity
$$\sum_{r=0}^{n} (-1)^r \binom{n}{r} = 0$$
for positive integers n.