MATH 301B SPRING 2011 PRACTICE FINAL

Write clearly, on separate paper. All questions carry equal weight.
You will receive credit for your five best answers.

(1) Prove or disprove the following statement:
A function $f : X \to Y$ is surjective if and only if there is a function $s : Y \to X$ such that $f \circ s = \mathrm{id}_X$.

(2) Let d be a positive integer. Show that the residue of an integer n modulo d is a unit of the monoid $(\mathbb{Z}/_d, \cdot, 1)$ if and only if n is coprime to d.

(3) Define

$$M = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in (\mathbb{Z}/_{11})^2_2 \;\middle|\; a_{21} = 0 \right\}.$$

(a) Show that M forms a submonoid of $\big((\mathbb{Z}/_{11})^2_2, \cdot, I_2\big)$.
(b) Determine the order $|M^*|$ of the group M^* of units of M.

(4) Suppose that H and K are subgroups of a group G.
(a) Give an example to show that HK need not be a subgroup of G.
(b) If $H \lhd G$ and $K \lhd G$, prove $HK \lhd G$.

(5) Let x and y be elements of a group G, with $xy = yx$. Let x have finite order a, and y have finite order b, with $\gcd(a, b) = 1$. Show that xy has order ab.

(6) Let I and J be ideals of a ring R. Show that

$$R/(I \cap J) \cong (R/I) \times (R/J)\,.$$