

MATH 301A FALL 2013 PRACTICE FINAL

Write clearly, on separate paper. All questions carry equal weight.
You will receive credit for your five best answers.

- (1) Prove or disprove the following statement:
A function $f : X \rightarrow Y$ is surjective if and only if there is a function $s : Y \rightarrow X$ such that $f \circ s = \text{id}_Y$.
- (2) For a positive integer d , define $\varphi(d) = |(\mathbb{Z}/d, \cdot, 1)^*|$.
(a) Show that $\varphi(p) = p - 1$ if p is prime.
(b) Show that $\varphi(mn) = \varphi(m)\varphi(n)$ if m and n are coprime.
- (3) Define

$$M = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in (\mathbb{Z}/13)_2^2 \mid a_{12} = 0 \right\}.$$

- (a) Show that M forms a submonoid of $((\mathbb{Z}/13)_2^2, \cdot, I_2)$.
(b) Determine the order $|M^*|$ of the group M^* of units of M .
- (4) Suppose that H and K are subgroups of a group G .
(a) Let L be the intersection of all the subgroups of G that contain both H and K . Show that L is a subgroup of G .
(b) Give an example to show that $H \cup K$ need not be a subgroup of G .
- (5) Let x and y be elements of a group G . Let x have finite order a , and y have finite order b , with $\gcd(a, b) = 1$.
(a) Show that xy has order ab if $xy = yx$.
(b) Give an example where $xy \neq yx$ and xy does not have order ab .
- (6) Prove the identity

$$\sum_{r=0}^{n/2} \binom{n}{2r} = \sum_{r=1}^{n/2} \binom{n}{2r-1}$$

for even natural numbers n .