MATH 201 SPRING 2024 GRADED HOMEWORK #3

Write clearly, on separate paper. All questions carry equal weight.

(1) Determine the limit of the sequence
$$\left\{ \frac{n^3 + n \cos^5(27n)}{10n^3 + 2n + 27} \right\}_{n \in \mathbb{N}},$$
carefully justifying your answer.

(2) Suppose that $f \colon \mathbb{R} \to \mathbb{R}; x \mapsto f(x)$ is a continuous function. For each natural number n, the function $f^{[n]} \colon \mathbb{R} \to \mathbb{R}; x \mapsto f^{[n]}(x)$ is defined recursively by
$$f^{[0]}(x) = x \,;$$
$$f^{[n+1]}(x) = f^{[n]}(f(x)) \,.$$

Give a careful proof by induction that the function $f^{[n]} \colon \mathbb{R} \to \mathbb{R}$ is continuous for each natural number n.

(3) Let $\{a_n\}_{n \in U}$ be a monotonically decreasing sequence which is not a Cauchy sequence. Show that, for every real number M, there is a tail $\{a_n\}_{n \in T}$ of $\{a_n\}_{n \in U}$ such that $\forall \, n \in T, \; a_n < M$.