MATH 201, MAY 1, 2020

This will be the last of our class blogs. I hope they've worked for you as we have wrestled with some subtle and sophisticated mathematics. It would have been hard for you to go back and locate any specific content you might have needed if it had been appearing in a video file, not to mention the excessive bandwidth that video files of the required resolution would have demanded.

One last time, let me remind you that the H3 assignment is due to be submitted through Canvas by 5 pm Central Daylight Time today. The instructions for preparing it are on the syllabus page. As before, the main thing to remember is that it must be in PDF format. The syllabus page has links to phone apps for scanning, if you need to write on paper and then scan.

Remember to name your submitted file as `[Yourlastname]M201H3`. For example, if I was submitting a file, it would be called `SmithM201H3`.

Once the submissions have been graded, I will post the solutions. Watch out on Canvas for the overall grading scheme. As has been the case all along, the median score will be set in the "B" range, and then we'll work up and down from there. You will need 50%, or 40 assignment points, to pass. But please see the Special Note after the Grading Policy on the Syllabus and Assignment page, on the open website or in Canvas.

Today's material isn't going to be "on the test"! From that point of view, once you've submitted your H3 assignment, you're done. But for the benefit of those of you who will be continuing to learn and use new mathematics, we should revisit the way that our textbook specified the set $\mathbb{R}$ of real numbers, as we saw it in the March 30 class blog:

Definition. The set $\mathbb{R}$ of *real numbers* forms the unique ordered field, containing $\mathbb{Q}$, that has the least upper bound property.

This specification of the real numbers is what we've had to work with all along, including when we needed to show that Cauchy sequences converge. For that task, we referred to the book's method, based on the sophisticated concepts of the $\limsup$ and $\liminf$.

However, there is an alternative way to prove the convergence result, avoiding $\limsup$ and $\liminf$. If we have a Cauchy sequence $\{x_n\}_{n\in U}$, we showed that it is bounded. In other words, there is a positive real number B such that all the sequence terms x_n lie in the finite closed interval $[-B, B]$.

Now consider the two halves $[-B, 0]$ and $[0, B]$ of $[-B, B]$. One of those two halves, say $[0, B]$, has to contain an infinite number of the x_n terms.

Now consider the two halves $[0, B/2]$ and $[B/2, B]$ of $[0, B]$. One of those two halves, say $[0, B/2]$, has to contain an infinite number of the x_n terms.

Now consider the two halves $[0, B/4]$ and $[B/4, B/2]$ of $[0, B/2]$. One of those two halves, say $[B/4, B/2]$, has to contain an infinite number of the x_n terms.

By this stage, you find yourself inevitably trapped into a Groundhog Day situation. You keep on halving, getting infinitely many of the x_n terms located in smaller and smaller intervals. Then the sequence has a limit, which lies in the intersection of this family of ever-shrinking intervals.

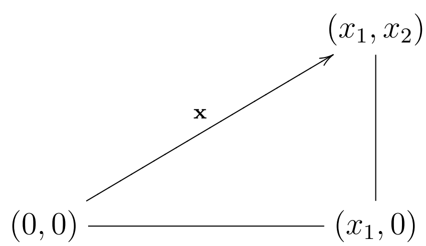
Incidentally, the homework question **2.4.5** that we worked out in the April 20 class blog takes care of the situation where, at any stage, we might have infinitely many terms in each of the two halves. In that situation, the limit is just the halfway point at that stage, by what we proved in the homework question.

We now have two ways to show that, in the set $\mathbb{R}$ of real numbers as specified in the book, Cauchy sequences converge. However, that is not the end of the story. The least upper bound property in the book's specification of $\mathbb{R}$ is not very satisfactory. It depends on the order structure of the real line $\mathbb{R}$, which you don't have available in any geometry of higher dimension, like the plane $\mathbb{R}^2$. And since the set $\mathbb{C}$ of complex numbers has the geometry of the plane $\mathbb{R}^2$ (people talk about "the complex plane"), the least upper bound property doesn't apply to complex numbers. Nevertheless, in advanced classes, you will need to think about sequences of complex numbers.

To deal with this issue, recall the book's definition of the absolute value function

$$|x| = \begin{cases} x & \text{if } x \geq 0; \\ -x & \text{if } x < 0 \end{cases}$$

using a case distinction based on the order structure of $\mathbb{R}$. On the other hand, the distance $|\mathbf{x}|$ of a two-dimensional position vector $\mathbf{x} = (x_1, x_2)$ from the origin in the plane $\mathbb{R}^2$



is $\sqrt{x_1^2 + x_2^2}$ by Pythagoras' Theorem. We took the one-dimensional version of this, namely $|x| = \sqrt{x^2}$, as our equivalent definition of the absolute value.

With this kind of measure $|\mathbf{x}|$ of the distance of a point $\mathbf{x}$ from the origin in a linear space (vector space) of any dimension, including $\mathbb{R}$ and $\mathbb{C}$, we can measure the distance between two points $\mathbf{x}$ and $\mathbf{y}$ as $|\mathbf{x}-\mathbf{y}|$. We can then declare a sequence $\{\mathbf{x}_n\}_{n\in U}$ of points in that space to be a **Cauchy sequence** if

$$\forall\, \varepsilon > 0\,,\ \exists\, M \in \mathbb{N}\,.\ \forall\, M \leq m, n \in U\,,\ |\mathbf{x}_m - \mathbf{x}_n| < \varepsilon\,.$$

This is just our familiar definition of a Cauchy sequence in $\mathbb{R}$, extended to the more general linear spaces.

Now, instead of having the convergence of Cauchy sequences as the conclusion of a theorem, we instead make it the content of the following definition.

Definition. A linear space is *complete* if every Cauchy sequence in it actually converges.

Together with what we have already seen, Exercise **2.4.3** in the book shows that for any ordered field that contains the set $\mathbb{Q}$ of rational numbers as a dense subset ($\forall\, x < y\,,\ \exists\, q \in \mathbb{Q}\,.\ x < q < y$), the property of completeness is equivalent to the least upper bound property. So finally, we arrive at a much more satisfactory specification of the set of real numbers:

Definition. The set $\mathbb{R}$ of *real numbers* forms the unique complete ordered field that contains $\mathbb{Q}$ as a dense subset.

We have come a long way from the naive picture of the real line that we saw in our very first class.