MATH 201, APRIL 27, 2020

We are entering the last week of our regular course. Traditionally, this time of the semester has meant preparation for final exams, which would then take place the following week. But because of the move online, ISU has now relaxed its guidelines for finals, and we will not be having a final exam as such. Instead, what would have been the third graded homework, originally announced as being due today, is morphing into the "Final Graded Homework".

It has already been posted on the Canvas (as Assignment H3) and the open website. It's due by 5 pm (Central Daylight Time) on Friday, 5/1. Instructions for preparing it are on the syllabus page. As before, the main thing to remember is that it must be in PDF format. The syllabus page has links to phone apps for scanning, if you need to write on paper and then scan. Overall, this process worked out well for the second graded homework.

Last time, we examined how the second half of the Comparison Test works, when we want to show that an unknown series diverges. The major application was to the harmonic series. This series provides a counterexample to the converse of the proposition that the summands of a convergent series tend to zero.

The Harmonic Series.

Proposition. If

$$\sum_{n=h}^{\infty} x_n$$

converges, then $\lim x_n = 0$.

Proof. A convergent series is a Cauchy series, so

$$\forall\, \varepsilon > 0\,,\ \exists\, M \in \mathbb{N}\,.\ \forall\, M < k\,,\ |x_k| = \left| \sum_{n=h}^{k} x_n - \sum_{n=h}^{k-1} x_n \right| < \varepsilon. \quad \square$$

The converse is false:

The Harmonic Series. This is

$$\sum_{n=1}^{\infty} \frac{1}{n}\,.$$

Although its summands $1/n$ tend to zero, the series diverges by the Comparison Test:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \dots \\ &\geq 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \dots \\ &= 1 + \frac{1}{2} + \left(\frac{2}{4}\right) + \left(\frac{4}{8}\right) + \dots \\ &= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots\,, \end{aligned}$$

which is unbounded, and thus divergent.

Homework problems

We will do the harder of the two problems, the one which needs a little processing to get the inequality to run in the right direction for our needs. Remember that if a fraction has a positive numerator, we make it smaller by increasing its denominator (and that ultimately, this behavior is a consequence of the ordered field properties of the set of real numbers).

2.5.3(a): Determine the convergencee or divergence of the series

$$\sum_{n=1}^{\infty} \frac{3}{9n+1} \,. \tag{1}$$

Solution. We have

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{3}{9n+1} &= \frac{3}{10} + \frac{3}{19} + \ldots + \frac{3}{9n+1} + \ldots \\ &\geq \frac{3}{10} + \frac{3}{20} + \ldots + \frac{3}{10n} + \ldots \\ &= \frac{3}{10} \left\{ \frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{n} + \ldots \right\} , \end{aligned}$$

which diverges as a nonzero multiple of the harmonic series. Thus, by the second part of the Comparison Test, the given series (1) also diverges.

There are infinitely many prime numbers

Although we are not having a final exam as such, we will still use this week to review some key aspects of what we have learned in our class. We started at a basic level, formulating the idea of a "mathematical statement", characterized by being either true or false (even if we don't actually know which of those two is correct). One such mathematical statement showed up in homework problem 4 from Section 2.9 of our first textbook:

> For every prime number p,
> there is another prime number q with $q > p$.

This statement is actually true. In fact, it is a famous and important result in number theory, known as **Euclid's Theorem**.

In order to be able to manipulate statements, especially complicated statements built from elementary ones, we studied Boolean algebra in parallel with the "ordinary" algebra of ordered fields. In both these kinds of algebras, the **distributive law** is an essential component. Thus multiplication distributes over sums in ordinary algebra, while in Boolean algebra, intersection distributes over union, and *vice versa*.

We saw a range of proof techniques, such as direct proof, handling distinct cases, contrapositive proof, proof by contradiction, and proof by induction. We used proof by contradiction to confirm that proof by induction works. While our first book treated "strong induction" as a separate technique to be justified and remembered, we learnt the more general trick of tweaking the induction hypothesis inside an induction proof. As an example of a result proved by induction, we showed:

> Every positive integer factorizes
> as a product of powers of prime numbers,

the **existence part** of the **Fundamental Theorem of Arithmetic**.

We have learned how to disprove false statements, particularly using a counterexample (as simple and concrete as possible!) to disprove a universally quantified statement. Sometimes we needed experiments to determine if a statement was likely true or false. For that purpose, we classified the statement as belonging to set theory, number theory, or real analysis, and then correspondingly used Venn diagrams, plugged in numbers, or reached for the graphing calculator. These classifications are actually all rather arbitrary. To gain a sense of the deep unity of mathematics, we will now use the geometric and harmonic series from real analysis to prove Euclid's Theorem from number theory.

Euclid's Theorem. There are infinitely many prime numbers.

Proof. Suppose that there are only finitely many prime numbers, say $p_1 < p_2 < \cdots < p_r$ in order, with $p_1 = 2$ and $p_2 = 3$. Bearing in mind that $p_i > 1$ for each $1 \leq i \leq r$, we can sum the geometric series

$$\frac{p_i}{p_i - 1} = \frac{1}{1 - \frac{1}{p_i}} = \frac{1}{p_i^0} + \frac{1}{p_i^1} + \frac{1}{p_i^2} + \ldots + \frac{1}{p_i^{e_i}} + \ldots\,, \tag{2}$$

since $0 < 1/p_i < 1$.

By the existence part of the Fundamental Theorem of Arithmetic, each positive integer n can be written in the form $n = p_1^{e_1} p_2^{e_2} \ldots p_r^{e_r}$ with natural number exponents e_i for $1 \leq i \leq r$.

We thus have

$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} + \ldots$$

$$= \frac{1}{p_1^0 p_2^0 p_3^0 \ldots p_r^0} + \frac{1}{p_1^1 p_2^0 p_3^0 \ldots p_r^0} + \frac{1}{p_1^0 p_2^1 p_3^0 \ldots p_r^0} + \frac{1}{p_1^2 p_2^0 p_3^0 \ldots p_r^0} + \ldots + \frac{1}{p_1^{e_1} p_2^{e_2} \ldots p_r^{e_r}} + \ldots \tag{3}$$

$$\leq \left(\frac{1}{p_i^0} + \frac{1}{p_i^1} + \ldots\right) \cdot \left(\frac{1}{p_2^0} + \frac{1}{p_2^1} + \ldots\right) \cdot \ldots \cdot \left(\frac{1}{p_r^0} + \frac{1}{p_r^1} + \ldots\right) \tag{4}$$

$$= \left(\frac{p_1}{p_1 - 1}\right) \cdot \left(\frac{p_2}{p_2 - 1}\right) \cdot \ldots \cdot \left(\frac{p_r}{p_r - 1}\right).$$

By the distributive law, each of the distinct terms that appear in the sum (3) preceding the inequality also appears in the product (4) on its right hand side. For the last step, we have used (2). The final product is clearly finite. We have the contradiction that the monotonically increasing partial sums of the harmonic series are bounded above, or in other words, the harmonic series converges. □