MATH 201, APRIL 20, 2020

The next assignment, an online test (or "quiz") due Wednesday 4/22 at 5 pm CDT (hard deadline!) has now been posted on Canvas. There are 14 questions. Each question has a mathematical statement. You have to identify whether the statement is true or false. We have covered all the material you need for the test. In particular, there are no questions on today's new material.

Here is the lesson summary from last time.

Cauchy sequences

Definition. The sequence $\{x_n\}_{n\in U}$ is *convergent* if

$$\exists\ L \in \mathbb{R}\,.\ \forall\ \varepsilon > 0\,,\ \exists\ M \in \mathbb{N}\,.\ \forall\ M \leq n \in U\,,\ |x_n - L| < \varepsilon\,.$$

— 4 quantifiers, compares terms against some limit L.

Definition. The sequence $\{x_n\}_{n\in U}$ is a *Cauchy sequence* if

$$\forall\ \varepsilon > 0\,,\ \exists\ M \in \mathbb{N}\,.\ \forall\ M \leq m, n \in U\,,\ |x_m - x_n| < \varepsilon\,.$$

— 3 quantifiers, compares terms against each other.

Proposition. A convergent sequence is a Cauchy sequence.

Proof estimate:

$$\begin{aligned} |x_m - x_n| &= |(x_m - L) + (L - x_n)| \\ &\leq |x_m - L| + |L - x_n| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,. \quad \square \end{aligned}$$

Proposition. A Cauchy sequence is bounded.

Proof. For $\{x_n\}_{n\in U}$, choose $M \in U$ so $\forall\ M \leq m, n \in U\,,\ |x_m - x_n| < 1$. Then $\forall\ k \in U\,,\ |x_k| \leq \max\big\{1 + |x_M|, \max\{|x_l| \mid M > l \in U\}\big\}$. $\square$

Theorem. Cauchy sequences converge.

Homework problems

2.4.1: Show directly from the definition that

$$\left\{\frac{n^2-1}{n^2}\right\}_{0<n\in\mathbb{N}}$$

is a Cauchy sequence.

Solution. We start by rewriting the sequence terms as

$$x_n = \frac{n^2-1}{n^2} = 1 - \frac{1}{n^2}\,.$$

Since the sequence $\{1/n^2\}$ converges to 0, we know that for a given tolerance ε, there is a (positive) cost M such that

$$\forall\ M \leq m, n \in \mathbb{N}\,,\ \frac{1}{n^2} < \frac{\varepsilon}{2}\,.$$

Thus, $\forall\ M \leq m, n \in \mathbb{N}\,,\ |x_m - x_n|$

$$= \left|\frac{1}{n^2} - \frac{1}{m^2}\right| \leq \left|\frac{1}{n^2}\right| + \left|\frac{1}{m^2}\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,,$$

verifying the Cauchy property.

We will not work out **2.4.4** in detail: It just uses a general version of our method for **2.4.1**, with y_n as the $\{2/n^2\}$ sequence there.

Now for the "challenge" problem.

2.4.5: Let $\{x_n\}$ be a Cauchy sequence such that

$$\forall\ M \in \mathbb{N}\,,\ \exists\ k \geq M\,.\ x_k < 0 \ \text{ and } \ \exists\ l \geq M\,.\ x_l > 0\,. \tag{1}$$

Show that $\lim x_n = 0$.

Solution. Consider a tolerance ε. By the Cauchy property, there is a cost M such that

$$\forall\ M \leq m, n\,,\ |x_m - x_n| < \frac{\varepsilon}{2}\,.$$

By the given property (1), we then have $x_l > 0$ and $x_k < 0$, with $k, l \geq M$, so $|x_| - x_k| < \varepsilon/2$, which means in particular that $|x_l| < \varepsilon/2$. Thus

$$\forall\ M \leq n\,,\ |x_n| = |(x_n - x_l) + x_l| \leq |x_n - x_l| + |x_l| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,,$$

as required for the convergence of $\{x_n\}$ to 0.

The new material: Series

In our 4/6 class, we started our work with sequences by considering successive approximations $s_0, s_1, s_2, \ldots$ to Euler's number, the base $e = 2.7182818284\ldots$ of natural logarithms:

Sequence element (partial sum)	Numerical value
s_0	$1.0000 = \frac{1}{0!}$
s_1	$2.0000 = \frac{1}{0!} + \frac{1}{1!}$
s_2	$2.5000 = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!}$
s_3	$2.6667 = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!}$
s_4	$2.7083 = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!}$
$\vdots$	$\vdots$

As you have probably noticed, this sequence was rather special (now reflected in our switch to the notation s_n and "partial sum"). In fact, as captured by the following definition, sequences of this type are called **series**, which come with their own special notations and terminology.

Definition. For $h \in \mathbb{N}$, a *series* or *infinite series*

$$\sum_{n=h}^{\infty} x_n \quad \text{or informally} \quad x_h + x_{h+1} + x_{h+2} + \ldots$$

means the sequence

$$\left\{ \sum_{n=h}^{k} x_n \right\}_{h \le k \in \mathbb{N}}$$

of *partial sums*

$$\sum_{n=h}^{k} x_n$$

of the *summands* x_n. We write

$$\sum_{n=h}^{\infty} x_n = L$$

to express that the sequence of partial sums converges to L.

Example: Euler's constant $e = \sum_{n=0}^{\infty} \frac{1}{n!} = 2.7182818284\ldots$.

Most of the sequence terminology carries over, so we have "convergent series," "bounded series," "divergent series," "Cauchy series," etc.

For today, we start working with series by explicitly finding a limit for the sequence of partial sums. This can only be done in certain special cases.

Geometric series:

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \qquad \text{for a “ratio” } r \text{ with } |r| < 1\,. \tag{2}$$

Here, consider the partial sum

$$s_k = \sum_{n=0}^{k} r^n = 1 + r + r^2 + \ldots + r^{k-1} + r^k$$

for a natural number k. Now always, $(1+r+\ldots+r^k)(1-r) = 1-r^{k+1}$, as may be checked informally by multiplying out and cancelling (try it!), or formally by induction on the natural number parameter k. Then, noting $1 - r \neq 0$ for $|r| < 1$, we have

$$s_k = \frac{1-r^{k+1}}{1-r} \to \frac{1}{1-r} \tag{3}$$

if $|r| < 1$, and that is exactly what the equation (2) is saying, according to the basic definition.

Note that we used Proposition 2.2.11(i) from the book here, setting $c = |r|$, to obtain $\lim |r|^{k+1} = 0$, and then the Squeezing Lemma applied to $-|r|^{k+1} \leq r^{k+1} \leq |r|^{k+1}$ to conclude that $\lim r^{k+1} = 0$ in (3).

Telescoping series: Given a convergent sequence $\{y_n\}_{h \leq n \in \mathbb{N}} \to y$,

$$\sum_{n=h}^{\infty} (y_n - y_{n+1}) = y_h - Y\,,$$

since

$$\begin{aligned}\sum_{n=h}^{k} (y_n - y_{n+1}) &= (y_h - y_{h+1}) + (y_{h+1} - y_{h+2}) + \ldots + (y_k - y_{k+1}) \\ &= y_h - y_{k+1} \to y_h - y\,.\end{aligned}$$

Example.

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \sum_{n=1}^{\infty} \left[\frac{1}{2n(n+1)} - \frac{1}{2(n+1)(n+2)}\right] \to \frac{1}{4}\,.$$

Here is the lesson summary.

Series

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means the sequence

$$\left\{\sum_{n=h}^{k} x_n\right\}_{h \leq k \in \mathbb{N}}$$

of *partial sums*

$$\sum_{n=h}^{k} x_n$$

of the *summands* x_n. Then write

$$\sum_{n=h}^{\infty} x_n = L$$

if the sequence of partial sums converges to L.

Example: Euler's constant $e = \sum_{n=0}^{\infty} \frac{1}{n!} = 2.7182818284\dots$.

Most of the sequence terminology carries over, so have "convergent series," "bounded series," "divergent series," "Cauchy series," etc.

Special series. Some series are easy to handle.

Geometric series:

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \quad \text{for "ratio" } r \text{ with } |r| < 1\,.$$

Telescoping series: Given a convergent sequence $\{y_n\}_{h \leq n \in \mathbb{N}} \to y$,

$$\sum_{n=h}^{\infty} (y_n - y_{n+1}) = y_h - Y$$

since

$$\sum_{n=h}^{k} (y_n - y_{n+1}) = (y_h - y_{h+1}) + (y_{h+1} - y_{h+2}) + \dots + (y_k - y_{k+1})$$
$$= y_h - y_{k+1} \to y_h - y\,.$$