MATH 201, APRIL 8, 2020

The grader finished all the grading of the second graded homework assignment shortly before the posting of this blog, so depending on how soon you read the blog, the grade lists on Canvas might not yet have been updated.

At any rate, it looks as if, on the whole, you've been doing a good job with the homework. The median score out of 12 was 10.

Following are some suggested solutions to the homework questions. They represent one way of tackling the problems, but other ways may also work.

Remember that technical communication is a major aspect of what we are learning in this class. So in a graded homework assignment, the grader is assessing how well you've communicated your mathematical message to them.

(1) Find the inverse of the function $f\colon \mathbb{R} \setminus \{-3\} \to \mathbb{R} \setminus \{2\}$ with
$$f(x) = \frac{2x+7}{x+3}\,.$$

Solution:

$$\begin{aligned} y = f(x) &= \frac{2x+7}{x+3} \\ \Leftrightarrow 2x+7 &= y(x+3) = yx + 3y \\ \Leftrightarrow x(2-y) &= 3y - 7 \\ \Leftrightarrow x &= \boxed{\frac{3y-7}{2-y} = f^{-1}(y)} \end{aligned}$$

(2) Prove or disprove:

Proposition. The set $D = \big\{\{m, n\} \subset \mathbb{N} \bigm| m \neq n\big\}$ of two-element subsets of $\mathbb{N}$ is countable.

Solution: TRUE.

Proof. Consider the relation $R = \big\{(m, n) \in \mathbb{N} \times \mathbb{N} \bigm| m < n\big\}$. The function

$$f : R \to D; (m, n) \mapsto \{m, n\}$$

is bijective, so $|R| = |D|$.

Now since $\mathbb{N}$ is countable, the set $\mathbb{N} \times \mathbb{N}$ is countable. Thus the infinite subset R of $\mathbb{N} \times \mathbb{N}$ is also countable, and $|R| = |\mathbb{N}|$. Hence $|D| = |\mathbb{N}|$. □

(3) Prove or disprove:

Proposition. The power set $\mathcal{P}(\mathbb{Z})$ of $\mathbb{Z}$ is countable.

Solution: FALSE.
By Russell's Paradox, $|\mathbb{Z}| < |\mathcal{P}(\mathbb{Z})|$, so $|\mathbb{N}| < |\mathcal{P}(\mathbb{Z})|$.

(4) Prove or disprove:

Proposition. Let $(S, <)$ be a (totally) ordered set. Let E be a nonempty subset of S for which $\inf E$ and $\sup E$ exist. Then $\inf E < \sup E$.

Solution: FALSE.
Consider $E = \{0\} \subseteq \mathbb{R}$. Then $\inf E = 0 = \sup E$.

Now we will recall the lesson summary from last time. We defined a sequence to be a real-valued function whoes domain is an infinite set of natural numbers, and explored the idea of convergence of a sequence to a limit. For a bounded increasing sequence, the limit always exists, as the supremum of the set of sequence values.

Sequences

Definition. A *sequence* $\{x_n\}_{n\in U}$ or $\{x_n\}$ is a function

$$U \to \mathbb{R}; n \mapsto x_n$$

whose domain is a countably infinite subset U of $\mathbb{N}$.

Definition. Let $\{x_n\}_{n\in U}$ be a sequence.

(1) The sequence is (*monotonic*) *increasing* if

$$m < n \in U \quad \Rightarrow \quad x_m \leq x_n \,.$$

(2) The sequence is (*monotonic*) *decreasing* if

$$m < n \in U \quad \Rightarrow \quad x_m \geq x_n \,.$$

(3) The sequence is *monotonic*
if it is monotonic increasing or decreasing.

Convergence of sequences. Let $\{x_n\}_{n\in U}$ be a sequence.

Definition. The sequence $\{x_n\}_{n\in U}$ *converges to a limit* L if

$$\forall\, \varepsilon > 0\,,\ \exists\, M \in \mathbb{N}\,.\ \forall\, M \leq n \in U\,,\ |x_n - L| < \varepsilon\,.$$

"tolerance" ↑ ↑ "cost"

Write $\lim_{n\to\infty} x_n = L$ or just $x_n \to L$.

Definition. A sequence $\{x_n\}_{n\in U}$ *converges* if

$$\exists\, L \in \mathbb{R}\,.\ \lim_{n\to\infty} x_n = L\,.$$

If not, the sequence *diverges*.

Propositions.

- Convergent sequences are bounded.
- A convergent sequence has a unique limit.
- Bounded monotonic increasing $\{x_n\}_{n\in U}$
  converges to $\sup\{x_n \mid n \in U\}$.
- Bounded monotonic decreasing $\{x_n\}_{n\in U}$
  converges to $\inf\{x_n \mid n \in U\}$.

Homework problems from the previous class

The problems give us a first taste of working with sequences.

2.1.1: Prove or disprove: The sequence $\{3n\}_{n\in\mathbb{N}}$ is bounded.

Proposition. The sequence $\{3n\}_{n\in\mathbb{N}}$ is not bounded.

Proof. For any real number B, the Archimedean property shows that there is a natural number n with $3n > B$. □

For the next problem, recall that convergent sequences are bounded.

2.1.2: Is the sequence $\{n\}_{n\in\mathbb{N}}$ convergent?

Solution. No, the sequence is not bounded, so it is not convergent.

For the final problem, we'll first play around a little on "scratch paper". As we move to the harder questions, where you have no idea what to do right away, this will become standard procedure. For positive integers m, n, we have $m^{-1/3} < n^{-1/3} \ \Leftrightarrow\ m^{1/3} > n^{1/3} \ \Leftrightarrow\ m^3 > n^3 \ \Leftrightarrow\ m > n$. Also, for $\varepsilon > 0$, we have $n^{-1/3} \leq \varepsilon \ \Leftrightarrow\ n^{1/3} \geq \varepsilon^{-1} \ \Leftrightarrow\ n \geq \varepsilon^{-3}$.

2.1.9: Show that the sequence $\{n^{-1/3}\}_{0<n\in\mathbb{N}}$ is monotone and bounded. Find the limit.

Solution. Note $m < n \Rightarrow m^3 < n^3 \Rightarrow m^{1/3} < n^{1/3} \Rightarrow m^{-1/3} > n^{-1/3}$, so the sequence is monotone decreasing. Further, $0 < n^{-1/3} < 1$ for positive integers n, so the sequence is bounded, with 0 as a lower bound. Finally,

$$\inf\{n^{-1/3} \mid 0 < n \in \mathbb{N}\} = 0,$$

since for $\varepsilon > 0$ there is a positive integer n with $n \geq \varepsilon^{-3}$, whence $n^{-1/3} \leq \varepsilon$, so ε no longer works as a lower bound. Thus the limit of the sequence is $\boxed{0}$.

[In our solution, note the use of the Archimedean property, with 1 as the tiny guy and ε^{-3} as the big bad bully. Now we're moving on to harder things, we no longer mention all that explicitly, but you will still need to think it through as you assemble your answers.]

The new material: Subsequences

With our definition of a sequence $\{x_n\}_{n\in U}$ as a real-valued function whose domain is an infinite subset U of $\mathbb{N}$, it is very easy to give our definition of a subsequence.

Definition. A *subsequence* $\{x_n\}_{n\in S}$ of a sequence $\{x_n\}_{n\in U}$ is a function

$$S \to \mathbb{R}; n \mapsto x_n$$

whose domain is a (countably) infinite subset S of U.

Compare this with Definition 2.1.16 in the book, where complicated double-level tags are needed.

One of the main issues with subsequences is the question of how the convergence properties of the original sequence and a subsequence are related. As a first step, we can prove the following very easily.

Proposition. If $\{x_n\}_{n\in U}$ converges to L,
then each subsequence $\{x_n\}_{n\in S}$ converges to L.

Proof. Given a tolerance $\varepsilon > 0$, let M be the corresponding cost for $\{x_n\}_{n\in U}$, so

$$\forall\ M \le n \in U\,,\ |x_n - L| < \varepsilon\,.$$

Then since $S \subseteq U$, we have

$$\forall\ M \le n \in S\,,\ |x_n - L| < \varepsilon\,.$$

Thus $\{x_n\}_{n\in S}$ converges to L. □

Note how we have completely avoided the induction that is needed for the book's proof of the corresponding result (Proposition 2.1.17).

In the opposite direction, we see that the convergence of a subsequence does not imply the convergence of the original sequence.

Example. A divergent sequence may have a convergent subsequence. Consider the sequence $\{x_n\}_{n\in\mathbb{N}}$ with

$$x_n = \begin{cases} n & \text{if } n \text{ is odd;} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Since the original sequence is unbounded, it does not converge. On the other hand, the subsequence $\{x_n\}_{n\in 2\mathbb{N}}$ is the constant sequence $\{0\}_{n\in 2\mathbb{N}}$, which converges to 0.

In contrast with general subsequences, whose convergence does not imply the convergence of the original sequence, there are special kinds of subsequences, called **tails**, whose convergence does extend to the full sequence.

Let $\{x_n\}_{n\in U}$ be a sequence.

Definition. A subsequence $\{x_n\}_{n\in T}$ of a sequence $\{x_n\}_{n\in U}$ is a *tail* if $U \smallsetminus T$ is finite.

Example. Here are some sample domains, $U = 3\mathbb{N}$ for the original sequence, S for a general kind of subsequence (missing all the even terms from U), and T for a tail, with $U \smallsetminus T = \{3, 6, 9\}$:

U:	0	3	6	9	12	15	18	21	24	...
S:		3		9		15		21		...
T:	0				12	15	18	21	24	...

Proposition. If a tail $\{x_n\}_{n\in T}$ of a sequence $\{x_n\}_{n\in U}$ converges to L, then $\{x_n\}_{n\in U}$ converges to L.

Proof. Let $K = \max(U \smallsetminus T)$, so $K < n \in U$ implies $n \in T$. Let M be the cost for making $\{x_n\}_{n\in T}$ match L with a tolerance ε, so

$$\forall\ M \le n \in T\,,\ |x_n - L| < \varepsilon\,. \tag{1}$$

Then $\max\{M, K+1\}$ is a cost for making $\{x_n\}_{n\in U}$ match L with a tolerance ε, since the condition $\max\{M, K+1\} \le n \in U$ makes $K < n$, putting n in T, and also makes $M < n$, so then $|x_n - L| < \varepsilon$ by (1). □

The proposition is summarized as the important observation that the convergence of a sequence only depends on what happens **ultimately**, and is not affected by what happens at the beginning.

Here is the lesson summary.

Subsequences

Definition. A *subsequence* $\{x_n\}_{n\in S}$ of a sequence $\{x_n\}_{n\in U}$ is a function

$$S \to \mathbb{R}; n \mapsto x_n$$

whose domain is a countably infinite subset S of U.

Proposition. If $\{x_n\}_{n\in U}$ converges to L,
then each subsequence $\{x_n\}_{n\in S}$ converges to L.

Fact: A divergent sequence may have a convergent subsequence.

Tails. Let $\{x_n\}_{n\in U}$ be a sequence.

Definition. A subsequence $\{x_n\}_{n\in T}$ of a sequence $\{x_n\}_{n\in U}$ is a *tail* if $U \smallsetminus T$ is finite.

Proposition. If a tail $\{x_n\}_{n\in T}$ of $\{x_n\}_{n\in U}$ converges to L,
then $\{x_n\}_{n\in U}$ converges to L.

Proof. Let $K = \max(U \smallsetminus T)$, so $K < n \in U$ implies $n \in T$.
Let M be the cost for making $\{x_n\}_{n\in T}$ match L with a tolerance ε,
so $\forall\ M \le n \in T\,,\ |x_n - L| < \varepsilon$.
Then $\max\{M, K+1\}$ is a cost for making $\{x_n\}_{n\in U}$ match L
with a tolerance ε,
since $\forall\ \max\{M, K+1\} \le n \in U\,,\ |x_n - L| < \varepsilon$. $\square$