MATH 201, APRIL 1, 2020

Here's the summary from last time. In the third definition, we finally nailed the set $\mathbb{R}$. The definition tells us what properties we can build on when we work with real numbers. The least upper bound property gives us the supremum $\sup E$ of a nonempty subset E that's bounded above, while the proposition gives $\inf E$ for nonempty E bounded below.

The set of real numbers

Definition. A set is a *field* if it contains 0 and 1, and carries addition, subtraction, multiplication, and division of nonzero elements, satisfying the usual rules (associativity, commutativity, etc.).

Definition. A field is an *ordered field* if it has a total order $<$ that "plays nice with the field," so satisfies the usual properties such as

$$a < b \quad \Rightarrow a + c < b + c \,, \quad \begin{cases} ac < bc & \text{if } c > 0 \\ ac = bc & \text{if } c = 0 \\ ac > bc & \text{if } c < 0 \end{cases} , \qquad \text{etc.}$$

Definition. The set $\mathbb{R}$ of *real numbers* forms the unique ordered field, containing $\mathbb{Q}$, that has the least upper bound property.

Subsets of the real numbers. If $E \subseteq \mathbb{R}$ and $c \in \mathbb{R}$, then

$$cE := \{cx \mid x \in E\} \,, \quad -E := \{-x \mid x \in E\} \,, \quad c+E := \{c+x \mid x \in E\} \,.$$

Proposition. Suppose $\emptyset \subset E \subset \mathbb{R}$ and E is bounded below.

(1) $-E$ is bounded above.
(2) E has a g.l.b. $\inf E = -\sup(-E)$.

Maxima and minima. Consider a nonempty finite subset E of $\mathbb{R}$. Then $\sup E$ (exists and) is often called the *maximum* $\max E \in E$.
Also $\inf E$ (exists and) is often called the *minimum* $\min E \in E$.
Use same notation any time $\sup E \in E$ or $\inf E \in E$.

Homework problems from the previous class

In the first problem, all we'll need from the set $\mathbb{R}$ is the ordered field property. Note how we implicitly negate "$x = 0$ and $y = 0$" for a contrapositive using de Morgan's Law to get "$x \neq 0$ or $y \neq 0$", and go straight for just one case with a "without loss of generality".

1.2.4: If $x^2 + y^2 = 0$ for $x, y \in \mathbb{R}$, then $x = y = 0$.

Proof. Without loss of generality, suppose $x \neq 0$. Then $x^2 > 0$. Now $y^2 \geq 0$, so $x^2 + y^2 > 0$. □

For the next question, recall the way we handled suprema and infima in the last lesson. Above and beyond that, we'll show one new technique: The use of the symbol ε ("epsilon") to denote a positive real number, typically thought of as a small number like 0.1 or 0.01 or 0.001. So as part of showing that a real number s works as the least upper bound for a set E, we'll show that $s - \varepsilon$ no longer works as an upper bound for E, with ε here standing for any positive real number you choose.

1.2.9: For nonempty bounded subsets A, B of $\mathbb{R}$, define

$$C = \{a + b \mid a \in A,\ b \in B\}\,.$$

Then $\sup C = \sup A + \sup B$.

Proof. We will show that $\sup A + \sup B$ satisfies the requirements to be the least upper bound for C.

(a) Consider a typical element $a + b$ of C, with $a \in A$ and $b \in B$. Then $a \leq \sup A$ and $b \leq \sup B$ imply $a + b \leq \sup A + \sup B$, so $\sup A + \sup B$ is an upper bound for C.

(b) For $\varepsilon > 0$, since $\sup A - \varepsilon/2$ is less than $\sup A$,

$$\exists\ a \in A\,.\ a > \sup A - \varepsilon/2\,. \tag{1}$$

Similarly,

$$\exists\ b \in B\,.\ b > \sup B - \varepsilon/2\,. \tag{2}$$

Taking a from (1) and b from (2), we then have

$$C \ni a + b > \sup A + \sup B - \varepsilon\,,$$

so $\sup A + \sup B - \varepsilon$ is not an upper bound for C. □

Note our use of the reversed membership sign ("$C \ni a + b$") in the final display. We don't want to use the symbol $\ni$ for "such that".

Today's new material: The Archimedean Property

We will now become acquainted with one of the most useful features of the set of real numbers: The Archimedean property. It is named for the Greek scientist Archimedes (check him out on Wikipedia!).

Archimedean property: $\forall\, t > 0\,,\ \forall\, b \in \mathbb{R}\,,\ \exists\, n \in \mathbb{N}\,.\ nt > b$.

Our notation is chosen carefully to reflect a schoolyard back story that is intended to help you remember and apply the property. Think of the positive number t as the "**t**iny little guy", and think of the real number b as the "**b**ig **b**ad **b**ully". So, how can the tiny little guy beat the big bad bully? By calling on some friends, the natural numbers, for help. The Archimedean property says that there is always a natural number n ready to step in and make $nt > b$.

When you get stuck with a problem, recall the Archimedean property. Decide what positive number will play the role of t (often, it will be our character ε), and what number will play the role of b.

OK, let's prove the Archimedean property.

Proof. If $b < 0$, take $n = 0$.
So now assume $b \geq 0$.
Suppose $\exists\, 0 < t \in \mathbb{R}\,.\ \forall\, n \in \mathbb{N}\,,\ nt \leq b\,.$
Thus $\forall\, n \in \mathbb{N}\,.\ n \leq b/t\,.$
Consider $m = \max\{k \in \mathbb{N} \mid k \leq b/t\}$.
Then $b/t < m + 1 \in \mathbb{N}$, a contradiction. □

As a "corollary", a follow-on result, we will state and prove a typical and very useful application of the Archimedean property. Note the use of ε. Because we're doing an infimum this time, we show that $0 + \varepsilon$ no longer functions as a lower bound. Today's lesson summary gives a slightly different formulation of the same proof.

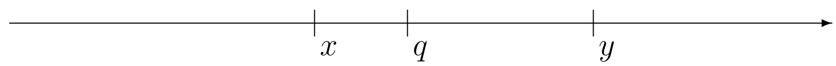
Corollary. Have $\inf\{1/n \mid 0 < n \in \mathbb{N}\} = 0$.

Proof. Note 0 is a lower bound for $\{1/n \mid 0 < n \in \mathbb{N}\}$.
Now consider $\varepsilon > 0$.
Set $t = \varepsilon$ and $b = 1$ in the Archimedean Property.
Thus $\exists\, 0 < n \in \mathbb{N}\,.\ n\varepsilon > 1$.
Then $\varepsilon > 1/n$, so ε is not a lower bound. □

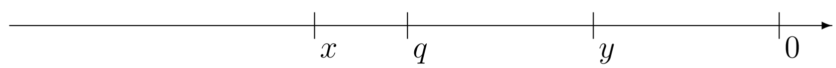
We will use this corollary a lot, not even quoting it.

Density of $\mathbb{Q}$ in $\mathbb{R}$. The Archimedean property may not look like much, but it has an amazing consequence. Although there are only countably many rational numbers, as opposed to uncountably many real numbers, nevertheless the rational numbers are spread out over the reals in such a way that every real number is as close as you like to a rational number.

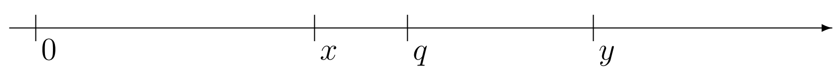
The actual result states that there is always a rational number q that manages to squeeze itself in between any pair $x < y$ of (distinct) real numbers. Here's the picture to keep in mind:



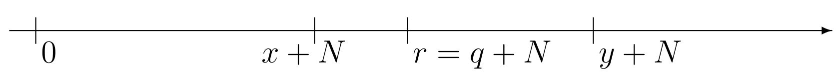
Now, that picture is actually located somewhere on the real line relative to zero. It could be around, or to the left, of zero:



In order to apply the Archimedean property on our proof, we'd like to have everything going on to the right of zero:



If it looks more like the previous picture, just shift everything way over to the right by adding some big natural number N. So instead of squeezing q between x and y, we'll use Archimedes to squeeze a rational number $r = q + N$ between $x + N$ and $y + N$:



We can always shift back to where we started by subtracting the natural number N off again. This is all taken care of in the proof on the next page by the "without loss of generality" phrase.

In the proof, note the assignment of roles to the tiny guy and the big bad bully for the Archimedean property. Also, note how we take the maximum of a finite subset of $\mathbb{R}$ (actually, of $\mathbb{N}$). That homework question we did for last time lets us do that.

Here's the actual result.

Density of $\mathbb{Q}$ in $\mathbb{R}$: $\forall\, x < y \in \mathbb{R}\,,\ \exists\, q \in \mathbb{Q}\,.\ x < q < y\,.$

Proof. Without loss of generality, assume $0 < x$.
Set $t = y - x$ and $b = 1$ in the Archimedean Property.
Thus $\exists\, 0 < n \in \mathbb{N}\,.\ n(y - x) > 1$.
Take $m = \max\{k \in \mathbb{N} \mid k < ny\}$.
Since $nx + 1 < ny$, we have $nx < m < ny$.
(Note $m \leq nx < ny - 1$ would imply $m + 1 < ny$, a contradiction.)
Take $q = m/n$, so $x < m/n < y$. □

As a sample application, we can now prove something stated previously without proof.

Proposition. The (totally) ordered set $\mathbb{Q}$ does not have the least upper bound property. In fact,

$$\sup\{x \in \mathbb{Q} \mid x^2 < 2\} = \sqrt{2}$$

in $\mathbb{R}$.

Proof. (a) First, note $0 < x$ and $x^2 < 2$ implies $x < \sqrt{2}$. Also, $x \leq 0$ implies $x < \sqrt{2}$. Thus $\sqrt{2}$ is an upper bound.

(b) Suppose $0 < \varepsilon < 1$, so $0 < \sqrt{2} - \varepsilon < \sqrt{2}$. By the density of $\mathbb{Q}$ in $\mathbb{R}$, there is a rational number q with $\sqrt{2} - \varepsilon < q < \sqrt{2}$. Then $q^2 < 2$, and $\sqrt{2} - \varepsilon$ is not an upper bound. □

Here is the lesson summary.

The Archimedean Property

Proposition. $\forall\, b \in \mathbb{R}\,,\ \forall\, 0 < t \in \mathbb{R}\,,\ \exists\, n \in \mathbb{N}\,.\ nt > b\,.$

Proof. If $b < 0$, take $n = 0$.
So now assume $b \geq 0$.
Suppose $\exists\, 0 < t \in \mathbb{R}\,.\ \forall\, n \in \mathbb{N}\,,\ nt \leq b\,.$
Thus $\forall\, n \in \mathbb{N}\,.\ n \leq b/t\,.$
Consider $m = \max\{k \in \mathbb{N} \mid k \leq b/t\}$.
Then $b/t < m + 1 \in \mathbb{N}$, a contradiction. $\square$

Corollary. Have $\inf\{1/n \mid 0 < n \in \mathbb{N}\} = 0$.

Proof. Note 0 is a lower bound for $\{1/n \mid 0 < n \in \mathbb{N}\}$.
Suppose $\inf\{1/n \mid 0 < n \in \mathbb{N}\} = \varepsilon > 0$.
Set $t = \varepsilon$ and $b = 1$ in the Archimedean Property.
Thus $\exists\, 0 < n \in \mathbb{N}\,.\ n\varepsilon > 1$.
Then $\varepsilon > 1/n$, so ε is not a lower bound — contradiction! $\square$

The density of $\mathbb{Q}$ in $\mathbb{R}$. This is a major consequence of the Archimedean Property.

Proposition. $\forall\, x < y \in \mathbb{R}\,,\ \exists\, q \in \mathbb{Q}\,.\ x < q < y\,.$

Proof. Without loss of generality, assume $0 < x$.
Set $t = y - x$ and $b = 1$ in the Archimedean Property.
Thus $\exists\, 0 < n \in \mathbb{N}\,.\ n(y - x) > 1$.
Take $m = \max\{k \in \mathbb{N} \mid k < ny\}$.
Since $nx + 1 < ny$, have $nx < m < ny$.
Take $q = m/n$, so $x < m/n < y$. $\square$