MATH 201, MARCH 30, 2020

Welcome to our second week of online classes. I hope you're getting into the routine. Reading the blog should be your substitute for going to the physical class. Don't be afraid to ask questions — just send me an e-mail!

The second graded homework has now been posted on the Canvas and the open website. It's due at 5 pm (Cnetral Daylight Tine) on Friday. Instructions for preparing it are on the syllabus page. The main thing to remember is that it must be in PDF format. Occasionally, people would send in homework as a jpeg photo, but the contrast was so low that it really made it hard for the grader to read. And you don't want to make life hard for the grader, especially in a class like ours where it's all about ease of communication.

The syllabus page has links to phone apps for scanning, if you need to write on paper and then scan. If you're using your favorite software, like LaTeX or Word, you can finish with a nice-looking PDF file from that. This blog is written in LaTeX, actually in the American Mathematical Society's AMSLaTeX version. I used to hate doing math with Word, because of its really clunky Equation Editor, but now they've replaced that with a simple TeX implementation. You key in "ALT" + "=" (regardless of Mac or PC, I believe) and then write the symbols using TeX commands. If you've got a little spare time, this may be the moment to start learning how to use some of the software. But don't let it obscure the mathematics for you.

The next page has the lesson summary from March 27. Using the new textbook, we're starting to build up towards our specification of $\mathbb{R}$. For the time being, we're just focusing on the strict order relation $<$, and not only in $\mathbb{R}$, either. The context is a (totally[1]) ordered set, which includes $\mathbb{Q}$ and $\mathbb{R}$. Mainly, right now, don't think aout doing any algebra, like adding or multiplying. We'll get to that in the new material, later.

[1]The textbook uses "ordered set" for what most people would think of as a "totally ordered set".

Bounds in totally ordered sets

Definition. A set S is (*totally*) *ordered* if it has a strict ordering relation $x < y$ such that the following two properties hold:

Trichotomy: For any two elements x, y of S, precisely one of the following three possibilities holds:

$$x < y \quad \text{or} \quad x = y \quad \text{or} \quad x > y \,.$$

Transitivity: For any three elements x, y, z of S, the implication

$$x < y \quad \text{and} \quad y < z \quad \text{implies} \quad x < z$$

holds.

Upper and lower bounds. Take subset E of totally ordered set S.

Definition.

- Element b of S is an *upper bound* for E if: $\forall\, x \in E\,,\ x \leq b\,.$
- Element b of S is a *lower bound* for E if: $\forall\, x \in E\,,\ b \leq x\,.$

Here, say E is respectively *bounded above* or *below* if such b exists.

Suprema and infima. Let E be a subset of a totally ordered set S.

Definition.

- An element l of S is the *supremum* or *least upper bound* (l.u.b.) $\sup E$ for E if:
  (a) l is an upper bound for E;
  (b) If b is an upper bound for E, then $l \leq b$.
- An element g of S is the *infimum* or *greatest lower bound* (g.l.b.) $\inf E$ for E if:
  (a) g is a lower bound for E;
  (b) If b is a lower bound for E, then $b \leq g$.

The least upper bound property. Let S be a totally ordered set.

Definition. Say S has the *least upper bound property* if:
whenever $\emptyset \subset E \subseteq S$ and E is bounded above, E has a l.u.b. in S.

- In $\mathbb{Q}$, $\{q \in \mathbb{Q} \mid q^2 < 2\}$ is bounded above, has no l.u.b. in $\mathbb{Q}$.
- In $\mathbb{R}$, $\{q \in \mathbb{Q} \mid q^2 < 2\}$ is bounded above, has l.u.b. $\sqrt{2}$ in $\mathbb{R}$.

Homework problems from the previous class

OK, time to try homework problems from last time. Here, we'll often only work one direction, say sup, as the other direction, say inf, will be similar (actually, "dual" as we say.) After you've seen the solution here for one direction, hide it away and try to do the other direction on your own.

1.1.2: For every nonempty finite subset A of a (totally) ordered set S, the least upper bound $\sup A$ exists as an element of A.

Proof. . . . by induction on $|A|$.

Ind. Basis: If $|A| = 1$, say $A = \{a_0\}$, then $\sup A = a_0 \in A$.

Ind. Step: Suppose the proposition holds for all sets B of positive cardinality n. Now consider a set $A = \{a_0, \ldots, a_{n-1}, a_n\}$ of cardinality $n + 1$. By the induction hypothesis,

$$\exists\ 0 \leq s < n\ .\ a_s = \sup\{a_0, \ldots, a_{n-1}\}\ .$$

Note $a_s \neq a_n$, since $|\{a_0, \ldots, a_{n-1}, a_n\}| = n + 1$.

Case I: $a_s < a_n$. Then $\sup A = a_n \in A$.

Case II: $a_n < a_s$. Then $\sup A = a_s \in A$. □

Three things to note here. First, see how the induction parameter became the cardinality of a set. Second, we carefully tweaked the induction hypothesis in order to make the induction step work. Third, trichotomy created the two cases in the induction step for us, once we'd dismissed the possibility of equality $a_s = a_n$ using the cardinality assumption.

Anticipating the new material from section 1.2.4 in the book, we will sometimes use the word *maximum*, and write $\max A$, whenever, as here, the supremum $\sup A$ of A actually lies in the set A. Dually, we'll write $\min A$ for $\inf A$ if $\inf A$ lies in A, and call it the *minimum*.

The second homework question we'll do goes in the opposite direction, assuming $\sup A \notin A$

1.1.6: In a (totally) ordered set S, let A be a nonempty subset, bounded above, such that $\sup A$ exists, but not as an element of A. Then A contains a countably infinite subset.

Remember recursive definitions (e.g, for x^n or $n!$) that are structured like induction proofs? Although our proof here is technically not an induction proof, it does use a recursive definition for the countably infinite subset of A.

Proof. A countably infinite subset $\{a_0 < a_1 < a_2 < \dots\}$ of A will be constructed recursively.

Recursion Basis: Since A is nonempty, it contains an element a_0.

Recursion Step: Suppose we have $\{a_0 < a_1 < a_2 < \cdots < a_n\} \subseteq A$. Since $a_n \in A$ and $a_n \leq \sup A \notin A$, we know $a_n < \sup A$ (trichotomy!). In particular, a_n is not an upper bound for A (since $\sup A$ is the least upper bound.) Thus, since the statement

$$\forall\ a \in A\,,\ a \leq a_n$$

does not hold, its negation

$$\exists\ a \in A\,.\ a > a_n$$

does. Choose a_{n+1} to be some such an element a that is strictly greater than the given element a_n. □

Note how trichotomy not only won us the strict inequality $a_n < \sup A$, it also helped us negate the punchline in the negation of the universally quantified statement.

Also, note how we are not allowed to just steal the dummy variable name "a" from the existentially quantified statement and declare a to be the next element in the recursive construction. Dummy variable names should only be understood and used within their context. They have no meaning to a reader outside that context.

Unlike the first two problems we just looked at, the two remaining problems did not need any extra proof ideas beyond what we have already done. We'll just go through one of them so you get into the right mood. Notice how it's all about working from the definitions, and using proof by contradiction (often implicitly).

1.1.9: In a (totally) ordered set S, suppose that A is a nonempty subset for which $\sup A$ exists. Suppose, for a certain subset B of A, that the following condition holds:

$$\forall\, x \in A\,,\ \exists\, y \in B\,.\ x \leq y\,. \tag{1}$$

Show $\sup B = \sup A$.

The proof works by showing that $\sup A$ satisfies the specification of $\sup B$.

Proof. (a) First, note that $\sup A$ is an upper bound for B, since B is a subset of A, and $\sup A \geq a$ for every element a of A.

(b) Now, we will show that $\sup A$ is the least upper bound for B. In other words, if we have an element s of S that is strictly less that $\sup A$, then it won't work as an upper bound for B.

Indeed, since $s < \sup A$, the element s is not an upper bound for A, so there is an element a of A with $s < a$. In the condition (1), take a as the element x of A. In the guise of y, the condition returns us an element b of B with $a \leq b$. Now by transitivity, $s < a \leq b$ implies $s < b$, meaning s is not an upper bound for B. □

The way we did the part (b) of the proof is typical of how we'll be checking the "least upper bound" property: show that any element of S that is less than the proposed least upper bound will no longer serve as an upper bound.

Later on, as we get good at all this, we will no longer be as explicit about using transitivity as we were in the proof of (b).

Today's new material: Specifying the set of real numbers

Finally, we have reached the point where we are ready to be precise about specifying the set $\mathbb{R}$ of real numbers. The set $\mathbb{R}$ carries two kinds of structure: algebra and order, and it is these two kinds of structure, together with the way that they interact, which characterize the set $\mathbb{R}$. So that's what we'll use in our specification.

ALGEBRA. You are already very familiar with the algebra structure of the set of real numbers, since we've been using it all through our class, either directly, or as a model for what may or may not happen with Boolean algebra, the algebra of sets or logical statements. Here is an informal definition, which is good enough for us. If you want more (gory) detail, see Definition 1.1.5 in the new textbook, or sign up for the Math 301 Abstract Algebra class.

Definition: A set with elements $0, 1$ is said to be a *field* if it has addition, subtraction, multiplication, and division by nonzero elements, where the usual rules like associativity of multiplication, distributivity of multiplication over addition, etc. are satisfied.

Examples: The sets $\mathbb{Q}$ and $\mathbb{R}$ are fields.

Examples: The sets $\mathbb{N}$ and $\mathbb{Z}$ are not fields. We can't always subtract in $\mathbb{N}$, e.g., $1 - 2 \notin \mathbb{N}$. We can't always do nonzero division in $\mathbb{Z}$, e.g., $1/2 \notin \mathbb{Z}$.

Example: The set $\{0, 1\}$ of binary digits (or "bits") forms both a field and a Boolean algebra.[2]. The usual real-number rules apply, along with $x - y = x + y$ and $1 + 1 = 0$.

ORDER. As a totally ordered set, $(\mathbb{R}, <)$ has the least upper bound property: Every nonempty subset A that is bounded above has a least upper bound $\sup A \in \mathbb{R}$.

Example: The totally ordered set $\{0 < 1\}$ of bits, with the usual Boolean algebra order, has the least upper bound property. This follows from our work on Exercise 1.1.2.

Example: The totally ordered set $(\mathbb{Q}, <)$ does not have the least upper bound property: $\{x \in \mathbb{Q} \mid x^2 < 2\}$ is bounded above by 5000, but has no least upper bound in $\mathbb{Q}$.

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[2]All you really need to know for that is $0 \wedge 1 = 0$ and $\neg 0 = 1$. In particular, $0 < 1$.

INTERACTION BETWEEN ALGEBRA AND ORDER.

Recall some of the rules we've been using all along as to how the algebra and order structure of $\mathbb{R}$ interact:

$$a < b \quad \Rightarrow a + c < b + c\,, \quad \begin{cases} ac < bc & \text{if } c > 0 \\ ac = bc & \text{if } c = 0 \\ ac > bc & \text{if } c < 0 \end{cases}, \qquad \text{etc.}$$

Proposition 1.1.8 in the book show how the usual rules follow whenever we have an *ordered field* in the following sense (Definition 1.1.7 in the book).

Definition. A field is an *ordered field* if it has a (total) order $<$ such that: $x < y \;\Rightarrow\; x + z < y + z$ and $x, y > 0 \;\Rightarrow\; xy > 0$ for all x, y, z.

Examples: The ordered sets $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$ form ordered fields.

Example: Bits do not form an ordered field, since while $0 < 1$, we have $0 + 1 = 1 > 0 = 1 + 1$.

So finally, here is our specification of the set of real numbers:

Definition. The set $\mathbb{R}$ of *real numbers* forms the unique ordered field, containing $\mathbb{Q}$, that has the least upper bound property.

The definition tells us what properties we are allowed to use when we are proving facts about real numbers. As we get into that, the following notation is useful. We already used it to write $2\mathbb{Z}$ for the set of even numbers and $1 + 2\mathbb{Z}$ for the set of odd numbers.

Subsets of the real numbers. If $E \subseteq \mathbb{R}$ and $c \in \mathbb{R}$, then

$$cE := \{cx \mid x \in E\}\,, \quad -E := \{-x \mid x \in E\}\,, \quad c{+}E := \{c{+}x \mid x \in E\}\,.$$

Now there's one thing about the specification of $\mathbb{R}$: While it gives you least upper bounds, it doesn't directly give you greatest lower bounds. The following (part of Prop. 1.2.6 in the book) does that for us.

Proposition. Suppose $\emptyset \subset E \subset \mathbb{R}$ and E is bounded below.

(1) $-E$ is bounded above.
(2) E has a g.l.b. $\inf E = -\sup(-E)$.

Here is the lesson summary.

The set of real numbers

Definition. A set is a *field* if it contains 0 and 1, and carries addition, subtraction, multiplication, and division of nonzero elements, satisfying the usual rules (associativity, commutativity, etc.).

Definition. A field is an *ordered field* if it has a total order $<$ that "plays nice with the field," so satisfies the usual properties such as

$$a < b \quad \Rightarrow a + c < b + c\,, \quad \begin{cases} ac < bc & \text{if } c > 0 \\ ac = bc & \text{if } c = 0 \\ ac > bc & \text{if } c < 0 \end{cases}, \qquad \text{etc.}$$

Definition. The set $\mathbb{R}$ of *real numbers* forms the unique ordered field, containing $\mathbb{Q}$, that has the least upper bound property.

Subsets of the real numbers. If $E \subseteq \mathbb{R}$ and $c \in \mathbb{R}$, then

$$cE := \{cx \mid x \in E\}\,, \quad -E := \{-x \mid x \in E\}\,, \quad c{+}E := \{c{+}x \mid x \in E\}\,.$$

Proposition. Suppose $\emptyset \subset E \subset \mathbb{R}$ and E is bounded below.

(1) $-E$ is bounded above.
(2) E has a g.l.b. $\inf E = -\sup(-E)$.

Maxima and minima. Consider a nonempty finite subset E of $\mathbb{R}$. Then $\sup E$ (exists and) is often called the *maximum* $\max E \in E$. Also $\inf E$ (exists and) is often called the *minimum* $\min E \in E$. Use same notation any time $\sup E \in E$ or $\inf E \in E$.