MATH 201, MARCH 27, 2020

Here is the lesson summary from March 25. For sets that may be infinite, we learned how to compare cardinalities, using the injective and surjective properties of functions. The most spectacular result was Russell's Paradox, using proof by contradiction and set-builder notation to show that the cardinality of any set is always strictly less than the cardinality of its power set.

Comparing cardinalities

For sets A, B, recall $|A| = |B|$ means there is a bijection $A \to B$.

Definition. Let A, B be sets.

(a) If there is a injective function $A \to B$,
but no surjective function $A \to B$, then $|A| < |B|$.

(b) If there is a injective function $A \to B$, then $|A| \leq |B|$.

Proposition. A subset of a countable set is countable.

Russell's Paradox. Recall the power set $\mathcal{P}(A)$ or 2^A of a set A.

Proposition. For any set A, have $|A| < \left|2^A\right|$.

Proof. Have injective function $A \to 2^A; a \mapsto \{a\}$.

Now suppose there is a surjective function $S\colon A \to 2^A$.
Consider $B = \{a \in A \mid a \notin S(a)\}$.
Since $S\colon A \to 2^A$ is surjective, have $b \in A$ with $S(b) = B$.

Case I: $b \in B$. Then $b \notin S(b) = B$, a contradiction.

Case II: $b \notin B$. Then $b \in S(b) = B$, a contradiction. $\square$

Homework problems from the previous class

Now we'll work some homework problems. By the way, you must wrestle with the assigned problems yourself, before we get to this point in the blog. Like they say, "Mathematics is not a spectator sport". Sitting in the bleachers won't make you an athlete.

14.2-2: Prove that the set $A = \{(m, n) \in \mathbb{N} \times \mathbb{N} \mid m \leq n\}$ is countably infinite.

Note that, in the language of book section 11-1, the set A is the relation $\leq$ on the set $\mathbb{N}$. If you were trying this exercise immediately after the March 23 class, you might have looked for a sweep pattern through the roster. Taking $\mathbb{N}$ here to mean the book's set of natural numbers (so really, positive integers), we could sweep left to right across each of a series of horizontal lines of increasing length: first $\{(1, 1)\}$, then $\{(1, 2), (2, 2)\}$, then $\{(1, 3), (2, 3), (3, 3)\}$, and so on.

But now we have a much better way to go. By definition, the set A is a relation ($\leq$) on the set $\mathbb{N}$, a subset of $\mathbb{N} \times \mathbb{N}$. Since $\mathbb{N}$ is countable, the set $\mathbb{N} \times \mathbb{N}$ is also countable. Therefore, the subset A is countable. It's also infinite, since it contains the infinite subset $\{(n, n) \mid n \in \mathbb{N}\}$. Done!

14.3-4: Prove or disprove: If $A \subseteq B \subseteq C$ and A and C are countably infinite, then B is countably infinite.

The claim is true. Here's the proof:

Proof. Since A is infinite, B is infinite. Since C is countable, B is countable. $\square$

14.3-8: Prove or disprove: The set $S = \{(a_0, a_1, a_2, \dots) \mid a_i \in \mathbb{Z}\}$ is countable.

This was the challenge problem. Kudos if you worked it! The claim is FALSE. So we have to prove its negation:

Proposition: The set $S = \{(a_0, a_1, a_2, \dots) \mid a_i \in \mathbb{Z}\}$ is uncountable.

A good way to tackle the proof would be a direct application of Cantor diagonalization. Just when anybody might have thought they'd got a nice countable list of all the sequences, say with

$$f(i) = (a_{i0}, a_{i1}, a_{i2}, \dots)$$

for each $i \in \mathbb{N}$, you could create the "diagonalized" sequence

$$d = (a_{00} + 1, a_{11} + 1, a_{22} + 1, \dots)$$

which, for each $i \in \mathbb{N}$, differs from $f(i)$ since $a_{ii} \neq a_{ii} + 1$.

Here's a formal write-up:

Proof. Suppose S is countable, say by a surjective function $f\colon \mathbb{N} \to S$ with

$$f(i) = (a_{i0}, a_{i1}, a_{i2}, \dots)$$

for each $i \in \mathbb{N}$. Now consider the sequence

$$d = (a_{00} + 1, a_{11} + 1, a_{22} + 1, \dots)\,.$$

Since $d \in S = f(\mathbb{N})$, there is a natural number i with

$$\begin{aligned} &(a_{i0} \qquad, a_{i1} \qquad, a_{i2} \qquad, \dots, a_{ii} \qquad, \dots) = f(i) \\ = d = &(a_{00} + 1, a_{11} + 1, a_{22} + 1, \dots, a_{ii} + 1, \dots)\,. \\ &\qquad\qquad\qquad\qquad \text{mismatch here} \uparrow \end{aligned}$$

But comparing the i-labeled slots in the claimed equality $f(i) = d$, we have $a_{ii} = a_{ii} + 1$, which is a contradiction. □

Today's new material, starting the second textbook

Now we move to the second textbook, *Basic Analysis*. If you need it, an online copy is linked off the Canvas and the open class website, This is the textbook that they use for the advanced Math 414 class, which sets out to give a careful, justified treatment of the limit processes that you use in calculus, like differentiation and integration.

Since *Basic Analysis* is an advanced textbook, it is not as relaxed and user-friendly as our first book. There are no solutions to odd-numbered problems.

- You will learn how to tackle the exercises by seeing what we do in the class blogs, and then by looking carefully through the examples that appear in the book sections listed in the reading assignments.
- You now have to fly solo in terms of knowing whether you have done a good job working a given problem. Here is where you will draw on all the techniques (e.g., using definitions, proof by contradiction, etc.) and logic that we have learnt from the first book. Check each step in your work. If you're not sure about it, break it up into smaller steps.

For our class, we will only be dealing with material selected from the first two chapters of *Basic Analysis*, and maybe one class dealing with continuous functions. We will be focusing on how to do proofs involving real numbers.

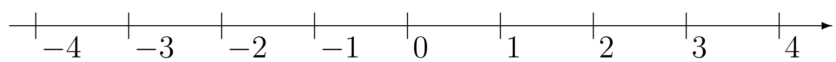
If you recall what we did in our very first class, we gave a tight roster-notation specification of the set

$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, 3, \dots\}$$

of *integers*, and a tight mixed-notation specification of the set

$$\mathbb{Q} = \left\{ \frac{m}{n} \;\middle|\; m, n \in \mathbb{Z}\,,\ n \neq 0 \right\}$$

of *rational numbers*. On the other hand, all we could do at that time was to present the intuitive picture



of the set $\mathbb{R}$ of real numbers, thinking of it as like the x-axis in a calculus-type graph or the display on your graphing calculator. So our first task now is going to involve building up a tight specification of the set of real numbers. It won't be that simple (which is why we didn't try it before), and it will take us a couple of classes.

A good starting point is to ask: How do we access a real number like $\sqrt{2} = 1.41421\dots$, given that right now we only have the set $\mathbb{Q}$ of rational numbers at our disposal?

Looking at the picture of $\mathbb{R}$, we see that the line is ordered from left to right. Intuitively, $x_1 < x_2$ if x_1 is to the left of x_2 in the picture.

Now, think about the set

$$E = \{x \in \mathbb{Q} \mid x^2 < 2\}$$

of rational numbers. The real number $\sqrt{2}$ lies to the **right** of every member x of E—it will be called an *upper bound* of E in $\mathbb{R}$—but it lies to the **left** of any other real number, like 1.5 or 1.42, which also happens to be an upper bound of E in $\mathbb{R}$.

In fact, $\sqrt{2}$ will be captured as the *least upper bound* of the subset E of $\mathbb{R}$. Being an "upper bound" stops $\sqrt{2}$ moving to the **left**, while having $\sqrt{2}$ less that any other upper bound prevents $\sqrt{2}$ from moving to the **right**. The two constraints match, and exactly locate the real number $\sqrt{2} = 1.41421\dots$ for us.

Now we're going to make our discussion precise. For today's class, we are only talking about order relationships $x_1 < x_2$, and disregarding any algebra (with the exception of the squaring in the definition of the subset E of $\mathbb{Q}$ or $\mathbb{R}$).

Definition. A set S is (*totally*) *ordered* if it has a strict ordering relation $x < y$ such that the following two properties hold:

Trichotomy: For any two elements x, y of S, precisely one of the following three possibilities holds:

$$x < y \quad \text{or} \quad x = y \quad \text{or} \quad x > y\,.$$

Transitivity: For any three elements x, y, z of S, the implication

$$x < y \quad \text{and} \quad y < z \quad \text{implies} \quad x < z$$

holds.

We will denote the set S, together with its ordering relation, as a pair $(S, <)$.

Notation. We will use the usual notations, like $y > x$ for $x < y$, or $x \leq y$ for "$x < y \quad \text{or} \quad x = y$",

Examples. The pairs $(\mathbb{N}, <)$, $(\mathbb{Z}, <)$, $(\mathbb{Q}, <)$, $(\mathbb{R}, <)$ all form (totally) ordered sets in the sense of the definition.

Example. The power set pair $\big(\mathcal{P}(\{0,1\}), \subset\big)$, with the proper subset relation $\subset$, does not form a totally ordered set. Trichotomy fails for $x = \{0\}$ and $y = \{1\}$: the two sets are not equal, and neither is properly contained in the other.

As we work with totally ordered sets, we will be working from the definition using the two key properties, trichotomy and transitivity. Trichotomy gives us useful case breakdowns. It is also great for doing the negation in a proof by contradiction. For example, $x \leq y$ negates to $x > y$, which we may also write as $y < x$.

Let E be a subset of a totally ordered set S.

Definition.

- Element b of S is an *upper bound* for E if: $\forall\ x \in E\,,\ x \leq b\,.$
- Element b of S is a *lower bound* for E if: $\forall\ x \in E\,,\ b \leq x\,.$

Here, say E is respectively *bounded above* or *below* if such b exists.

Example. Let's consider our subset $E = \{x \in \mathbb{Q} \mid x^2 < 2\}$ of $\mathbb{Q}$ or $\mathbb{R}$. Then 10 is an upper bound, so the set E is bounded above. The real number $-\pi$ is a lower bound of E in $\mathbb{R}$, but not in $\mathbb{Q}$, since it's not even an element of $\mathbb{Q}$. However, -200 is a lower bound for E in $\mathbb{Q}$. Thus E is bounded below in both $\mathbb{Q}$ and $\mathbb{R}$.

Example. Every subset of $\mathbb{N}$ is bounded below, with 0 as a lower bound.

Suprema and infima. These two Latin words are fancy names for least upper bounds and greatest lower bounds. The English phrases are good for helping you remember the two properties (a) and (b) in the definition below. But the Latin words give us two widely used mathematical notations: $\sup E$ and $\inf E$.

Let E be a subset of a totally ordered set S.

Definition.

- An element l of S is the *supremum* or *least upper bound* (l.u.b.) $\sup E$ for E if:
  (a) l is an upper bound for E;
  (b) If b is an upper bound for E, then $l \leq b$.
- An element g of S is the *infimum* or *greatest lower bound* (g.l.b.) $\inf E$ for E if:
  (a) g is a lower bound for E;
  (b) If b is a lower bound for E, then $b \leq g$.

Some people read l.u.b. and g.l.b. as "lub" and "glub".

As we work with this extremely important definition, we will be checking out the two properties (a) and (b). The proof of the following proposition will show how that might go.

Proposition. Consider a nonempty subset E of a totally ordered set S. Suppose that E has least upper bounds s_1 and s_2 in S. Then $s_1 = s_2$.

Proof. We will prove $s_1 \leq s_2$. Here, think of s_1 as a least upper bound, and just think of s_2 as an upper bound, which we're allowed to do by part (a) of the definition of least upper bound as it applies to s_2.

Now, since s_1 is a least upper bound, and s_2 is an upper bound, part (b) of the definition of least upper bound, as it applies to s_1, gives us $s_1 \leq s_2$.

A similar proof shows $s_2 \leq s_1$. Then the desired equation $s_1 = s_2$ follows by trichotomy: it's the only possible case left in the conjunction of $s_1 \leq s_2$ and $s_2 \leq s_1$. □

Now that the proposition has told us that a set E has a unique least upper bound, if it has one at all, we will write $\sup E$ for that least upper bound (supremum).

A similar proposition, flipping around the role of upper and lower bounds, shows that infima are unique, if they exist at all. Write $\inf E$ in that case. In future, we'll generally do one direction of a proposition like this, just dismissing the flip as being similar. Sometimes, the flip will be called the "dual".

A slicker approach would be to consider the ordered set $(S, >)$ in its own right: An upper bound for E in $(S, >)$ is a lower bound for E in $(S, <)$, and so on. This observation can also be described as "duality".

The final definition gives a property that an entire totally ordered set S may or may not possess.

Definition. A totally ordered ser S has the *least upper bound property* if every nonempty subset E that is bounded above actually has a least upper bound in S.

For the time being, we will just state the follwoing results. Later, we'll be able to prove them:

- In $\mathbb{Q}$, $\{q \in \mathbb{Q} \mid q^2 < 2\}$ is bounded above, has no l.u.b. in $\mathbb{Q}$.
- In $\mathbb{R}$, $\{q \in \mathbb{Q} \mid q^2 < 2\}$ is bounded above, has l.u.b. $\sqrt{2}$ in $\mathbb{R}$.

Here is the lesson summary.

Bounds in totally ordered sets

Definition. A set S is (*totally*) *ordered* if it has a strict ordering relation $x < y$ such that the following two properties hold:

Trichotomy: For any two elements x, y of S, precisely one of the following three possibilities holds:

$$x < y \quad \text{or} \quad x = y \quad \text{or} \quad x > y\,.$$

Transitivity: For any three elements x, y, z of S, the implication

$$x < y \quad \text{and} \quad y < z \quad \text{implies} \quad x < z$$

holds.

Upper and lower bounds. Take subset E of totally ordered set S.

Definition.

- Element b of S is an *upper bound* for E if: $\forall\ x \in E\,,\ x \leq b\,.$
- Element b of S is a *lower bound* for E if: $\forall\ x \in E\,,\ b \leq x\,.$

Here, say E is respectively *bounded above* or *below* if such b exists.

Suprema and infima. Let E be a subset of a totally ordered set S.

Definition.

- An element l of S is the *supremum* or *least upper bound* (l.u.b.) $\sup E$ for E if:
  (a) l is an upper bound for E;
  (b) If b is an upper bound for E, then $l \leq b$.
- An element g of S is the *infimum* or *greatest lower bound* (g.l.b.) $\inf E$ for E if:
  (a) g is a lower bound for E;
  (b) If b is a lower bound for E, then $b \leq g$.

The least upper bound property. Let S be a totally ordered set.

Definition. Say S has the *least upper bound property* if:
whenever $\emptyset \subset E \subseteq S$ and E is bounded above, E has a l.u.b. in S.

- In $\mathbb{Q}$, $\{q \in \mathbb{Q} \mid q^2 < 2\}$ is bounded above, has no l.u.b. in $\mathbb{Q}$.
- In $\mathbb{R}$, $\{q \in \mathbb{Q} \mid q^2 < 2\}$ is bounded above, has l.u.b. $\sqrt{2}$ in $\mathbb{R}$.