MATH 201, MARCH 25, 2020

Here is the lesson summary from March 23. For infinite sets, we learned the difference between being countable (so countably infinite), like $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$, or uncountable, like $\mathbb{R}$. Roster notation sweeps showed that $\mathbb{Z}$ and $\mathbb{Q}$ are counatble, while Cantor diagonalization showed that $\mathbb{R}$ is uncountable.

Countable and uncountable sets

Definition. Let A be a non-empty set.

(a) If there is a surjective function $f\colon \mathbb{N} \to A$,
i.e., A can be written in roster notation as $A = \{a_0, a_1, a_2, \dots\}$,
then A is *countable*.

(b) Otherwise, A is *uncountable*.

(c) If $|\mathbb{N}| = |A|$, then A is *countably infinite*.

Examples. Finite sets, $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ are countable.
The latter three are counatbly infinite. Finite sets are not.

Theorem. A, B countable $\Rightarrow A \cup B$, $A \times B$ countable.

Cantor diagonalization. The set $\mathbb{R}$ is uncountable.
Suppose $f\colon \mathbb{N} \to \mathbb{R}$ is surjective.

$$\begin{aligned}
f(0) &= n_0.\mathbf{a}_{00}a_{01}a_{02}a_{03}\dots\\
f(1) &= n_1.a_{10}\mathbf{a}_{11}a_{12}a_{13}\dots\\
f(2) &= n_2.a_{20}a_{21}\mathbf{a}_{22}a_{23}\dots\\
f(3) &= n_3.a_{30}a_{31}a_{32}\mathbf{a}_{33}\dots\\
&\ \vdots\\
\text{Choose } x = {}&\ 0.\mathbf{b}_{00}\mathbf{b}_{11}\mathbf{b}_{22}\mathbf{b}_{33}\dots
\end{aligned}$$

If $\forall\ i \in \mathbb{N}$, $\mathbf{a}_{ii} \neq \mathbf{b}_{ii}$, then $x \notin f(\mathbb{N})$, a contradiction.

Homework problems from the previous class

Let's work two of the homework problems. They will illustrate the important proof-writing technique of "**quoting theorems**". We'll use this technique a lot, especially when we switch to the second textbook.

14.2-4: Prove: The set $\mathbb{R} \smallsetminus \mathbb{Q}$ of irrational numbers is uncountable.

Let's try a proof by contradiction:

Proof. Suppose $\mathbb{R} \smallsetminus \mathbb{Q}$ is countable. Then $\mathbb{R}$, as the union

$$\mathbb{R} = (\mathbb{R} \smallsetminus \mathbb{Q}) \cup \mathbb{Q}$$

of the countable sets $\mathbb{R} \smallsetminus \mathbb{Q}$ and $\mathbb{Q}$, is countable. This contradicts $\mathbb{R}$ being uncountable. □

That worked quite easily, given the theorems we have from the lesson summary. The key point to be aware of here is how we recalled those theorems, or, as mathematicians say, how we "quoted" them. We can't say "by Theorem 14.6" or "by Theorem 14.2", since those numbers only make sense for a particular edition of a particular textbook. So our argument for $(\mathbb{R} \smallsetminus \mathbb{Q}) \cup \mathbb{Q}$ being countable was to point out to the reader the actual reason, built in to the statement of the theorem, which is that both of the two uniands $\mathbb{R} \smallsetminus \mathbb{Q}$ and $\mathbb{Q}$ are countable (within the context of the proof by contradiction).

14.2-6: Prove or disprove: There exists a bijection $f \colon \mathbb{Q} \to \mathbb{R}$.

Looking at the claim, it would mean $|\mathbb{Q}| = |\mathbb{R}|$. Informally, that's like saying that $\mathbb{Q}$ and $\mathbb{R}$ have the same (infinite) number of elements. So we conclude that the result is FALSE. Now, how do we disprove it? Remember our disproof techniques, which depend on the nature of the claim we're disproving. Here, we are disproving an existence statement, which we can parse in the following form:

$$\exists\ f \colon \mathbb{Q} \to \mathbb{R}\,.\ f \text{ is bijective.}$$

Given that, we can run our "flip the quantifiers, negate the punchline" routine to set up the negation that we have to present and prove:

$$\forall\ f \colon \mathbb{Q} \to \mathbb{R}\,,\ \ f \text{ is not bijective.}$$

Proposition. A function $f \colon \mathbb{Q} \to \mathbb{R}$ cannot be bijective.

Proof. Suppose $f \colon \mathbb{Q} \to \mathbb{R}$ is bijective. Then $|\mathbb{Q}| = |\mathbb{R}|$. We know $|\mathbb{N}| = |\mathbb{Q}|$, and so $|\mathbb{N}| = |\mathbb{R}|$, contradicting the uncountability of $\mathbb{R}$. □

Today's new material

This will be our last work from the first, easy textbook, covering stuff from Section 14.3, before we move to the second, more advanced textbook. Up to this point, we have been working with **equalities between cardinalities**. The proof of our proposition at the end of the previous page is a very good example of that. Now, we will consider **inequalities between cardinalities** in the following basic definition. Note how we're not bothering to name the functions involved. That's another useful flexibility in our powerful function notation.

Definition. Let A and B be sets.

(a) If there is a injective function $A \to B$,
but no surjective function $A \to B$, then $|A| < |B|$.

(b) If there is a injective function $A \to B$, then $|A| \leq |B|$.

Example. If B is any non-empty set, then $0 = |\emptyset| < |B|$.

Proposition. If $A \subseteq B$, then $|A| \leq |B|$.

Proof. Consider the so-called *insertion* function $j \colon A \to B; a \mapsto a$. It is certainly injective, since $j(a_1) = j(a_2)$ for $a_1, a_2 \in A$ just means $a_1 = a_2$. Thus $|A| \leq |B|$ by part (b) of the definition. $\square$

Insertion functions $j \colon A \to B; a \mapsto a$, embedding a set A as a subset of another set B, are useful functions to be aware of in general, not just for the proof of the proposition.

Example. Note that the claim "If $A \subset B$, then $|A| < |B|$" is false. As a counterexample, we might consider the proper subset $\mathbb{N}$ of $\mathbb{Z}$, so $\mathbb{N} \subset \mathbb{Z}$. Then we still have $|\mathbb{N}| = |\mathbb{Z}|$.

Make sure you read Theorems 14.8 and 14.9 from the book section.

Theorem 14.8 should help you do Exercise 14.2-2, if you weren't able to do it with a roster sweep the first time around. That's why I passed on discussing the exercise in today's blog.

Now, let's have a little fun to close out the stuff from the first textbook. (Yes, I know, a math professor's idea of fun!) Suppose you were asked to prove the following proposition:

Proposition. The inequality $n < 2^n$ holds for all natural numbers n.

How would you set about proving it? Induction, right? Just so we don't forget how to do induction proofs, as we'll need plenty of them when we get to the new textbook, let's do this one, following our standard pattern so we're not reinventing the wheel.

Proof. ...by induction on n.

Induction Basis: For $n = 0$, we have $n = 0 < 1 = 2^0 = 2^n$.

Induction Step: Suppose $n < 2^n$. Then

$$n + 1 < 2^n + 1 \leq 2^n + 2^n = 2 \times 2^n = 2^{n+1}$$

as required. $\square$

Now, recall the definition of the *power set* 2^A or

$$\mathcal{P}(A) = \{S \mid S \subseteq A\}$$

of a set A. In other words, the set of all subsets of A. The book uses the displayed notation. The reason for the exponential notation 2^A is that

$$2^{|A|} = \left|2^A\right| \tag{1}$$

for any finite set A. Thinking of the natural number (i.e., a finite set cardinality!) $|A|$ as n, we can reformulate our proposition as the following:

Proposition. The inequality $|A| < \left|2^A\right|$ holds for all finite sets A.

So right now, we have the induction proof for this proposition. But on the next page, we will prove the proposition for **all** sets, not just for finite sets! And there will be no induction, just a really sneaky use of set-builder notation inside a proof by contradiction. Buckle up!

The proof technique for the following result is known as **Russell's paradox**. In the proof, we will revert to using $\mathcal{P}(A)$ for the power set of a set A.

Proposition. The inequality $|A| < \left|2^A\right|$ holds for all sets A.

Proof. Consider a set A. By part (a) of our definition, we have two things to prove.

(a): There is an injective function

$$A \to \mathcal{P}(A); a \mapsto \{a\}.$$

Indeed, for $a_1, a_2 \in A$, we have $\{a_1\} = \{a_2\} \Rightarrow a_1 = a_2$.

(b): There is no surjective function $A \to \mathcal{P}(A)$.

Suppose there was a surjective function $S\colon A \to \mathcal{P}(A); a \mapsto S(a)$.
Consider $B = \{a \in A \mid a \notin S(a)\}$.
Since the function $S\colon A \to \mathcal{P}(A)$ is surjective, there is an element $b \in A$ with $S(b) = B$.

Case I: $b \in B$. Then $b \notin S(b) = B$, a contradiction.

Case II: $b \notin B$. Then $b \in S(b) = B$, a contradiction. $\square$

As a consequence of this proposition, there are infinitely many infinite cardinalities

$$|\mathbb{N}| < |\mathcal{P}(\mathbb{N}0| < |\mathcal{P}(\mathcal{P}(\mathbb{N}0)| < |\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{N}0))| < \dots$$

And as usual, today's blog concludes with the lesson summary.

Comparing cardinalities

For sets A, B, recall $|A| = |B|$ means there is a bijection $A \to B$.

Definition. Let A, B be sets.

(a) If there is a injective function $A \to B$,
but no surjective function $A \to B$, then $|A| < |B|$.
(b) If there is a injective function $A \to B$, then $|A| \leq |B|$.

Proposition. A subset of a countable set is countable.

Russell's Paradox. Recall the power set $\mathcal{P}(A)$ or 2^A of a set A.

Proposition. For any set A, have $|A| < \left|2^A\right|$.

Proof. Have injective function $A \to 2^A; a \mapsto \{a\}$.

Now suppose there is a surjective function $S\colon A \to 2^A$.
Consider $B = \{a \in A \mid a \notin S(a)\}$.
Since $S\colon A \to 2^A$ is surjective, have $b \in A$ with $S(b) = B$.

Case I: $b \in B$. Then $b \notin S(b) = B$, a contradiction.

Case II: $b \notin B$. Then $b \in S(b) = B$, a contradiction. $\square$