MATH 201, MARCH 23, 2020

Welcome back from the spring break! I hope you were all able to recharge your batteries, ready for the second half of the semester. It will be a challenge for everybody at Iowa State, adapting to a changed way of doing things. Our goal now will be to complete the class as best we can under the circumstances. Hopefully, by the end of the semester, you will be ready to apply the mathematical ideas, thought processes, and communication techniques that we learn in Math 201 to the more advanced classes in your program, be they mathematics, economics, engineering, statistics, or whatever.

Just before the spring break, the plan was for us to have interactive video classes, using the Conferences feature in Canvas. The idea was to recreate the regular classroom experience as closely as possible in an online format. Well, that was then, and this is now! ISU can no longer use Conferences, and has not recommended any interactive format, or "synchronous" as they call it. Apparently it takes way more bandwidth than they think is going to be available, now it seems like almost the entire white-collar workforce has gone remote, and the internet providers are struggling to keep up.

In this environment, and bearing in mind the sometimes intense and technical nature of the content of our class, we will be moving to our new blog format. It should be reliably available on whatever platform you have been using so far to access the class materials. Reading the thrice-weekly blog will be the substitute for our previous class hours. As in the first half of the semester, you should be attempting all the assigned "ongoing" homework from one class in your own time (one to two hours) before the next class. You may e-mail me with questions, and in the first half of each new blog, we will explore selected homework problems from the previous class, before moving on to the new material. The lesson summaries that we had in the first half-semester will now appear within the blogs. The blogs are linked after the class assignments, in the place where the lesson summaries were linked.

Here is the lesson summary from March 13, where we learned three equivalent ways to think about a function $f\colon A \to B; x \mapsto f(x)$, having both the existence and uniqueness conditions on solutions to $y = f(x)$ for any $y \in B$, with the solution $x \in A$. Recall that $|A| = |B|$ is our first step towards being able to count infinite sets:

$$\text{Bijective} \equiv \text{invertible} \equiv \text{isomorphism}$$

Definition. On a set A, have the *identity function* $\mathrm{id}_A\colon A \to A; x \mapsto x$.

Definition. Function $f\colon A \to B; x \mapsto f(x)$ is *invertible* if there is a function $g\colon B \to A; y \mapsto g(y)$ such that $\forall\ x \in A\,,\ g(f(x)) = x$ and also $\forall\ y \in B\,,\ f(g(y)) = y$, i.e., $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$.

Note $g\colon B \to A$ is unique, the *inverse* $f^{-1}\colon B \to A$ of invertible f.

Definition. Function $f\colon A \to B; x \mapsto f(x)$ is *bijective* if both injective and surjective.

Bijective ≡ invertible: $\quad f(x) = y \quad \Leftrightarrow \quad x = f^{-1}(y)$.
(existence and uniqueness of the solution $x \in A$ to $f(x) = y$ for $y \in B$)

Isomorphic sets. Say sets A and B are *isomorphic* whenever there is a bijective function $f\colon A \to B$. Then write $A \cong B$ or $|A| = |B|$ — same (possibly infinite) *cardinality*.

Definition. Function $f\colon A \to B; x \mapsto f(x)$ is a (*set*) *isomorphism* if it is bijective.

Note $|\mathbb{N}| = |A|$ means A is infinite, and can be written in roster notation as $A = \{a_0, a_1, a_2, \dots\}$.

Homework problems from the previous class

Let's take a look at a couple of typical homework problems.

12.2-9: Prove $f\colon \mathbb{R} \setminus \{2\} \to \mathbb{R} \setminus \{5\}; x \mapsto \frac{5x+1}{x-2}$ is bijective.

12.5-2: Find the inverse of the bijective function from 12.2-9.

In a single "one-line proof" of elementary algebra, we will do both questions together. In fact, we'll find the inverse by equation-solving. The existence of the inverse then shows that f is *invertible*, and thus *bijective* (since those two concepts mean the exact same thing).

Here we go:

$$\begin{aligned} & y = f(x) = \frac{5x+1}{x-2} \\ \Leftrightarrow\ & 5x+1 = y(x-2) = yx - 2y \\ \Leftrightarrow\ & x(5-y) = -2y-1 \\ \Leftrightarrow\ & x = \boxed{\frac{-2y-1}{5-y} = f^{-1}(y)} \end{aligned}$$

That's all you need!

Note the "$\Leftrightarrow$" signs that we're using as connectives. A collection of disjoint equations does not make sense to a reader. (Of course, on our private "scratch paper", we're free to do whatever we like.)

Also, note that we were able to divide by $5 - y$ at the last $\Leftrightarrow$, since y, being from the codomain $\mathbb{R} \setminus \{5\}$, is not allowed to equal 5. This is a typical example of why the codomain is such an important part of our function specifications, going beyond the much vaguer approach to functions that you get in calculus.

Finally, note that we kept the y as the symbol for typical elements of the set B (the domain for f and the codomain for f^{-1}). Not only is there no *need* to switch variables around (like the book suggests), it would be a bad thing to do for two reasons. We're keeping x for the elements of A, and rewriting could be a source of error.

We'll now start doing some infinite counting.

14.1-8: For $S = \{x \in \mathbb{R} \mid \sin x = 1\}$, show $|\mathbb{Z}| = |S|$.

To amswer this, we need to begin with a little trigonometry to get a handle on the set S. Once we have S rewritten with mixed notation involving integers, the problem becomes easy. If you've forgotten your trigonometry, don't tell anyone, just graph $\sin x$ with your graphing calculator.

Soln. Note

$$x \in S \Leftrightarrow \sin x = 1 \Leftrightarrow x = \frac{\pi}{2} + 2n\pi \text{ for some } n \in \mathbb{Z}\,.$$

Thus

$$S = \left\{ \frac{\pi}{2} + 2n\pi \;\middle|\; n \in \mathbb{Z} \right\},$$

and

$$f \colon \mathbb{Z} \to S; n \mapsto \frac{\pi}{2} + 2n\pi$$

is the required bijection.

Now, if your worst enemy insists on not believing that f is a bijection, remind them that

$$g \colon S \to \mathbb{Z}; x \mapsto \frac{x - \frac{\pi}{2}}{2\pi}$$

is the inverse f^{-1} to f. In other words, recalling the composition of functions, and the definition of "invertibile" from the lesson summary, that means $g(f(n)) = n$ for $n \in \mathbb{Z}$ and $f(g(x)) = x$ for $x \in S$. Check it out, at least mentally!

As we progress to higher levels, we will start taking bigger steps in our arguments. That's OK, if you always make sure you could fill in if you had to. When you're communicating mathematics, be aware of who your audience is, and tailor your level of detail to their level of knowledge and familiarity with the topic. This is the ART of proof-writing, which goes along with the SCIENCE of mathematics.

Today's new material

Right, let's move on to the new material for today. We'll cover stuff from Sections 14.1 and 14.2 from the third edition of our first textbook. Here's the basic definition, with some very important terminology:

Definition. Let A be a non-empty set.
(a) If there is a surjective function $f : \mathbb{N} \to A$,
i.e., A can be written in roster notation as $A = \{a_0, a_1, a_2, \dots\}$,
then A is *countable*.
(b) Otherwise, A is *uncountable*.
(c) If $|\mathbb{N}| = |A|$, then A is *countably infinite*.

Example. A finite, non-empty set is countable.

We'll also add that the empty set is countable, by convention, even though it can't be the codomain of any function from a non-empty domain like $\mathbb{N}$, let alone a surjective function, because there's no place for the function values that would have to be there.

Warning: Some people (they could be instructors from your more advanced classes!) tend to reserve "countable" for what we are calling "countably infinite" in part (c) of the definition.

In connection with (c) from the definition, note that any infinite, countable set is countably infinite. That's where the name comes from!

In connection with (a) from the definition, once we have the set A in the roster notation, we can formally define the surjective

$$f : \mathbb{N} \to A; n \mapsto a_n$$

as our "counting" function. Often, and necessarily so if A is finite, an element of A may show up as an a_r for infinitely many natural numbers r. But that doesn't matter. It's just the way the set braces $\{$ and $\}$ work, if you recall from our very first class.

Let's see how to use this roster notation idea in some proofs. Also, read the book sections carefully!

Proposition. If A and B are countable, then so is $A \cup B$.

Proof. Take
$$A = \{a_0, a_1, a_2, \dots\}$$
and
$$B = \{b_0, b_1, b_2, \dots\}.$$
Then
$$A \cup B = \{a_0, b_0, a_1, b_1, a_2, b_2, \dots\}$$
as required. $\square$

Note the interleaving of the elements from a and b in the roster notatiuon for $A \cup B$.

Proposition. If A and B are countable, then so is $A \times B$.

Proof. Take
$$A = \{a_0, a_1, a_2, \dots\}$$
and
$$B = \{b_0, b_1, b_2, \dots\}.$$
Then
$$A \times B = \left\{ \begin{array}{cccc} \vdots & \vdots & \vdots & \\ c_5 = (a_0, b_2), & c_8 = (a_0, b_0), & c_{12} = (a_2, b_3), & \dots \\ c_2 = (a_0, b_1), & c_4 = (a_1, b_1), & c_7 = (a_2, b_1), & \dots \\ c_0 = (a_0, b_0), & c_1 = (a_1.b_0), & c_3 = (a_2, b_0), & \dots \end{array} \right\}$$
as required. $\square$

In the roster notation array for $A \times B$, trace out the elements $c_0, c_1, c_2, c_3, \dots$ in order, so you can see the pattern being used to sweep out the two-dimensional display of $A \times B$. The sweep pattern here is different from the one used in the book in Figure 14.2. Our pattern is more systematic. You could even construct a complicated formula for our $c_t = (a_r, b_s)$, giving $t = t(r, s)$ in terms of the arguments r and s. For example,
$$t(r, 0) = \binom{r+1}{2}$$
for $r > 0$. But that is not a question we need to get into right now.

The set $\mathbb{Z}$ is countable, indeed $|\mathbb{N}| = |\mathbb{Z}|$. Here's one possible line-up:

$$\begin{array}{lccccccc} \mathbb{N}: & \dots & 4 & 2 & 0 & 1 & 3 & \dots \\ \mathbb{Z}: & \dots & -2 & -1 & 0 & 1 & 2 & \dots \end{array}$$

You could convert this into a formula for a bijective function

$$f\colon \mathbb{N} \to \mathbb{Z}; n \mapsto \begin{cases} -\frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases}$$

if you wanted, but again, that's not our focus. In Exercise 14.2-2 for next time, just come up with a "sweep pattern". I won't discuss that exercise.

For the proof that $\mathbb{Q}$ is countable, see Theorem 14.4 in the book. The main argument is a sweep of a two-dimensional array, like we had showing that $A \times B$ is countable when A and B are.

The deepest and most important result we'll consider today is the **Cantor diagonalization**, which really means the key argument in the proof by contradiction that we'll give for the theorem below.

Theorem. The set $\mathbb{R}$ is uncountable.

Proof. Suppose there is a surjective function $f\colon \mathbb{N} \to \mathbb{R}$, so that $\mathbb{R}$ is countable. Consider the respective function values of the natural number arguments, real numbers written out as decimal expansions, like $\pi = 3.14159\dots$, etc.

$$\begin{aligned} f(0) &= n_0.\mathbf{a}_{00}a_{01}a_{02}a_{03}\dots \\ f(1) &= n_1.a_{10}\mathbf{a}_{11}a_{12}a_{13}\dots \\ f(2) &= n_2.a_{20}a_{21}\mathbf{a}_{22}a_{23}\dots \\ f(3) &= n_3.a_{30}a_{31}a_{32}\mathbf{a}_{33}\dots \\ &\vdots \end{aligned}$$

Now choose a real number $x = \ 0.\mathbf{b}_{00}\mathbf{b}_{11}\mathbf{b}_{22}\mathbf{b}_{33}\dots$

such that, for each natural number i, the digit b_{ii} appearing in the decimal expansion of x **differs** from the corresponding digit $\mathbf{a}_{ii}$ in the decimal expansion of $f(i)$. This already means that $x \neq f(i)$. Thus $x \notin f(\mathbb{N})$, a contradiction to f being surjective. $\square$

Here's a quick illustration of the argument. Suppose

$$\begin{aligned} f(0) &= \sqrt{2} = 1.\mathbf{4}142\ldots \\ f(1) &= e = 2.7\mathbf{1}82\ldots \\ f(2) &= \pi = 3.14\mathbf{1}5\ldots \\ &\vdots \end{aligned}$$

Now, we could choose $x = 0.527\ldots$. Then $x \neq \sqrt{2}$ since $\sqrt{2}$ has 4 in its first decimal place, while x has 5 there. Next, $x \neq e$ since e has 1 in its second decimal place, while x has 2 there. And so on.

This clever diagonalization[1] argument goes back to Georg Cantor (check him out on Wikipedia!), but it turns out to have all sorts of modern applications. For example, it implies that there are functions $p\colon \mathbb{N} \to \mathbb{N}$ for which you cannot write a computer program. (There are only countably many programs, but uncountably many functions.)

The next page has the summary of today's lesson.

[1]The *diagonal* is formed by the bolded digits in the arrays.

Countable and uncountable sets

Definition. Let A be a non-empty set.

(a) If there is a surjective function $f\colon \mathbb{N} \to A$,
i.e., A can be written in roster notation as $A = \{a_0, a_1, a_2, \ldots\}$,
then A is *countable*.
(b) Otherwise, A is *uncountable*.
(c) If $|\mathbb{N}| = |A|$, then A is *countably infinite*.

Examples. Finite sets, $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ are countable.
The latter three are counatbly infinite. Finite sets are not.

Theorem. A, B countable $\Rightarrow A \cup B$, $A \times B$ countable.

Cantor diagonalization. The set $\mathbb{R}$ is uncountable.
Suppose $f\colon \mathbb{N} \to \mathbb{R}$ is surjective.

$$\begin{aligned} f(0) &= n_0.\mathbf{a}_{00}a_{01}a_{02}a_{03}\ldots \\ f(1) &= n_1.a_{10}\mathbf{a}_{11}a_{12}a_{13}\ldots \\ f(2) &= n_2.a_{20}a_{21}\mathbf{a}_{22}a_{23}\ldots \\ f(3) &= n_3.a_{30}a_{31}a_{32}\mathbf{a}_{33}\ldots \\ &\vdots \\ \text{Choose } x = &\ \ 0.\mathbf{b}_{00}\mathbf{b}_{11}\mathbf{b}_{22}\mathbf{b}_{33}\ldots \end{aligned}$$

If $\forall\, i \in \mathbb{N}\,,\ \mathbf{a}_{ii} \neq \mathbf{b}_{ii}$, then $x \notin f(\mathbb{N})$, a contradiction.